PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 36

General review

Schedule for PHY 711 presentations
Monday Dec. 4, 2017

Time	Presenter	Title
9:00-9:15 AM	Ali Daraei	Rheology of fibrin fibers
9:17-9:32 AM		
9:34-9:49 AM	Yan Li	Computational study of normal modes for one-dimensional mass-spring oscillating systems

Wednesday Dec. 6, 2017

Time	Presenter	Title
9:00-9:15 AM	Matthew Waldrip	Vibrational modes of a 3D system
9:17-9:32 AM	Nouf Alharbi	Vibrational Modes of Carbon Dioxide
9:34-9:49 AM	Kevin Roebuck	Equations of motion of the Foucault pendulum

Friday Dec. 8, 2017

Time	Presenter	Title
9:00-9:15 AM	Haardik Pandey	Normal modes of vibrations for water molecule
9:17-9:32 AM	Ellie Alipour	
9:34-9:49 AM	TJ Colvin	Quasiharmonic analysis of biomolecules

Equations describing fluid mechanics

Mass density: $\rho(\mathbf{r},t)$

Fluid velocity: $\mathbf{v}(\mathbf{r},t)$

Fluid pressure: $p(\mathbf{r},t)$

External force per unit mass: $f(\mathbf{r},t)$

Viscosity: η, ζ

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Navier-Stokes equation

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Continuity condition

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Special case -- incompressible fluid: ρ=constant and $\nabla\cdot\mathbf{v} = 0$

Studied solutions in the linear limit

Another special case -- incompressible fluid: ρ=constant and $\nabla \cdot \mathbf{v} = 0$

For the special non-linear case of surface waves leading to the Korteweg-de Vries equation

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0$$

with the solution:

$$\eta(u) = \eta(x-ct) = \eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Another special case -- adiabatic ideal gas

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$ ⟸ adiabatic conditions

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Sound waves in the linear limit:

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Heat diffusivity equation

Temperature distribution: $\frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$

$\kappa \equiv \frac{k_{th}}{\rho c_p}$ thermal diffusivity

Solution to initial value case in a simple domain:

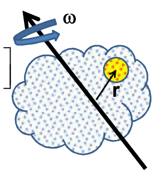
$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with} \quad G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Rigid body motion

$$T = \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \ddot{\mathbf{I}} \cdot \boldsymbol{\omega}$$



Moment of inertia tensor :

$$\ddot{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

Matrix notation :

$$\ddot{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Descriptions of rotation about a given origin

For general coordinate system

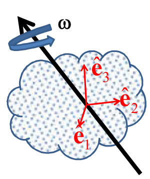
$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

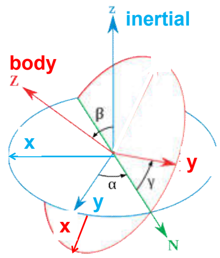
$$\ddot{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

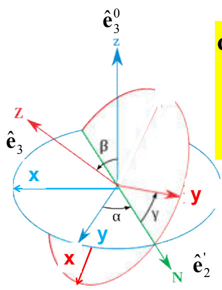


Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Rotational kinetic energy

$$\begin{aligned}T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) &= \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2 \\ &= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2 \\ &+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2 \\ &+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2\end{aligned}$$

If $I_1 = I_2$:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

Mathematical methods

- Sturm-Liouville equations and special functions
- Eigenfunctions
- Green's functions
- Fourier transforms
- Laplace transforms
- Contour integrals

Contour integrals using the Residue Theorem:

Another result of this analysis is the Residue Theorm which states that if the complex function $g(z)$ has poles at a finite number of points z_p within a region C but is otherwise analytic, the contour integral can be avaluated according to

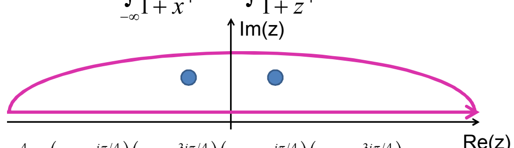
$$\oint_C g(z)dz = 2\pi i \sum_p Res(g_p), \tag{6}$$

where the residue is given by

$$Res(g_p) \equiv \lim_{z\rightarrow z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_p)^m g(z)\right) \right\}, \tag{7}$$

where m denotes the order of the pole.

Example: $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \oint \frac{z^2}{1+z^4}dz$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\text{Res}\left(z_p = e^{i\pi/4}\right) + \text{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

$$\text{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \text{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{\sqrt{2}}$$

Cauchy integral theorem for analytic function *f(z)*:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'.$$

Kramers-Kronig relationships

For $f(x) = a(x) + ib(x)$

$$\Rightarrow a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

Please reference Lecture 15 for a review of the first portion of the semester.