PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 4:
Reading: Chapter 1 F&W

1. Summary of previous discussion of scattering theory; transformation between lab and center of mass frames
2. Scattering theory in the center of mass frame; Calculation of the scattering cross section
3. Cross section for Rutherford scattering

PHY 711 Classical Mechanics and Mathematical Methods

MWF 9 AM-9:50 AM | OPL 107 | http://www.wfu.edu/~natalie/f17phy711/

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017				
6	Fri, 9/08/2017				
7	Mon, 9/11/2017				
8	Wed, 9/13/2017				

PHY 711 -- Assignment #2

Aug. 30, 2017

Read Chapter 1 in **Fetter & Walecka**.

1. In class, we "derived" the differential cross section for the scattering of two hard spheres of mutual radius *D* in the center of mass frame. Find the differential cross section for this system in the lab frame in which m_{target} is initially at rest and evaluate the expression for the following cases.
   a. $m_1/m_{target}=0.1$
   b. $m_1/m_{target}=1$
   c. $m_1/m_{target}=1000$

PHY 711 -- Assignment #3

Sept. 4, 2017

Continue reading Chapter 1 in **Fetter & Walecka**.

1. Work Problem #1.15 at the end of Chapter 1 in **Fetter and Walecka**.

Scattering theory:

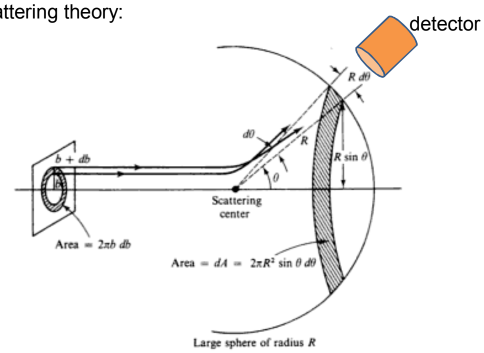
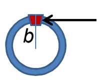


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

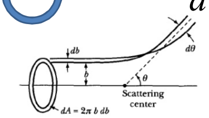
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$=$ Area of incident beam that is scattered into detector at angle θ



$$d\sigma = d\phi b db$$

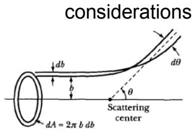
$$d\Omega = d\phi \sin\theta d\theta$$

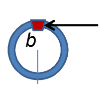


$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations





$d\varphi b db$

Figure from Marion & Thorton, Classical Dynamics

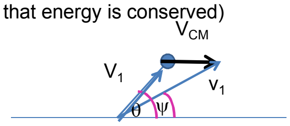
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)

Lab (θ) vs CM (ψ)

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$



$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

where : $\tan\psi = \dfrac{\sin\theta}{\cos\theta + m_1/m_2}$

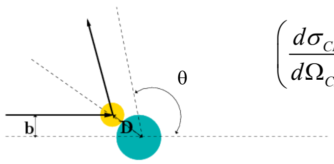
Example: suppose $m_1 = m_2$

In this case : $\tan\psi = \dfrac{\sin\theta}{\cos\theta + 1} \quad \Rightarrow \psi = \dfrac{\theta}{2}$

note that $0 \le \psi \le \dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Example of hard spheres



$$\left(\frac{d\sigma_{CM}}{d\Omega_{CM}}\right) = \frac{D^2}{4}$$

Cross section in lab frame when $m_1 = m_2$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} \quad \Rightarrow \psi = \frac{\theta}{2} \quad \Rightarrow \text{note that} \quad 0 \le \psi \le \frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right) \cdot 4\cos\psi = D^2 \cos\psi$$

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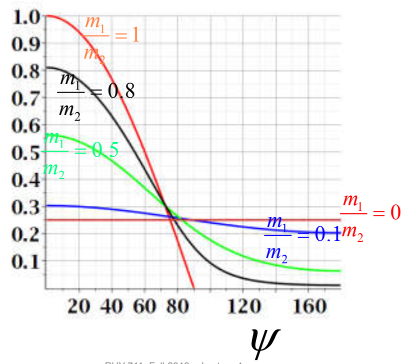
Scattering cross section for hard sphere in lab frame for various mass ratios:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)$$



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For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \psi(\theta)$$

Maple syntax:

```
> plot( { [psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
    = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
    sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
    thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
    orange])
```

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Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

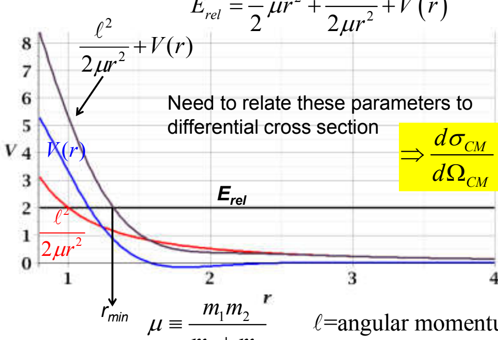
Hard sphere: $V(r) = \begin{cases} \infty & r \le D \\ 0 & r > D \end{cases}$ X

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ←

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



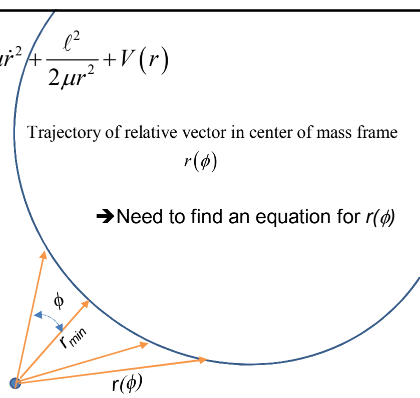
Need to relate these parameters to differential cross section $\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$

$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$ ℓ=angular momentum

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame $r(\phi)$

➔Need to find an equation for $r(\phi)$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for *r(φ)* ⇔ *φ(r)*:

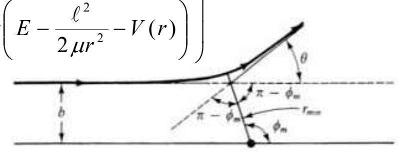
From: $E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

v_∞ →



Special values at large separation ($r \to \infty$):

$$\ell = \mu\left|\mathbf{r} \times \mathbf{v}\right|_{r\to\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \varphi_{\max}(b,E) = \varphi\left(r \to \infty\right) - \varphi\left(r = r_{\min}\right)$$

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$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

where :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

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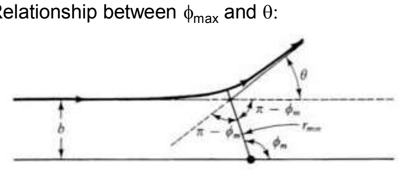
Relationship between $\phi_{\max}$ and θ:



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

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$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

Scattering angle equation:

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

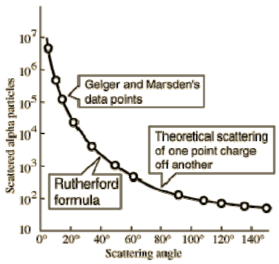
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2}$$