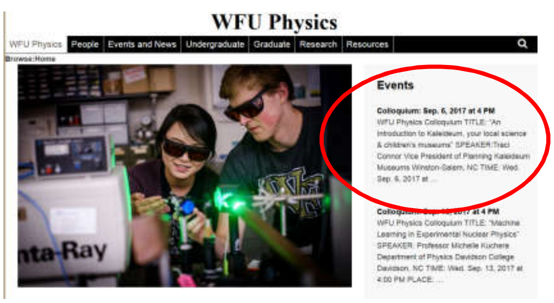
PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 5:
Reading: Chapter 2 of Fetter & Walecka

Physics described in accelerated coordinate frames

1. Linear acceleration
2. Angular acceleration
3. Foucault pendulum



WFU Physics Colloquium

TITLE: "An Introduction to Kaleideum, your local science & children's museums"

SPEAKER: Traci Connor
Vice President of Planning
Kaleideum Museums
Winston-Salem, NC

TIME: Wed. Sep. 6, 2017 at 4:00 PM

PLACE: George P. William:
101)

ABSTRACT

Traci Connor, VP of Planning at Kaleideum, specializes in exhibit development and evaluating the impact of exhibits. She will be providing an overview of science and children's museums including how scientists are partnering with these types of institution across the nation. She will also give us a virtual tour of our local science and children's museums, and a sneak preview into what they are planning now that they have merged into one institution, Kaleideum.

MWF 9 AM-9:50 AM | OPL 107 | http://www.wfu.edu/~natalie/f17phy711/

Instructor: Natalie Holzwarth | Phone:758-5510 | Office:300 OPL | e-mail:natalie@wfu.edu

Course schedule

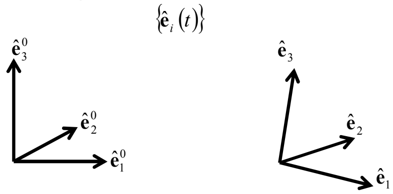
(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in an non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017				
7	Mon, 9/11/2017				
8	Wed, 9/13/2017				
9	Fri, 9/15/2017				

Physics in accelerated reference frames

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\{\hat{\mathbf{e}}_i^0\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\{\hat{\mathbf{e}}_i(t)\}$



Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

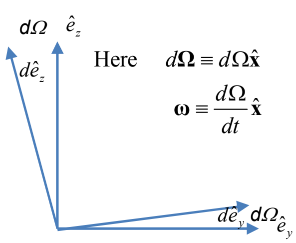
$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define}: \left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Properties of the frame motion (rotation only):



Here $d\mathbf{\Omega} \equiv d\Omega\hat{\mathbf{x}}$

$$\boldsymbol{\omega} \equiv \frac{d\Omega}{dt}\hat{\mathbf{x}}$$

$$d\hat{e}_y = d\Omega\hat{e}_z$$

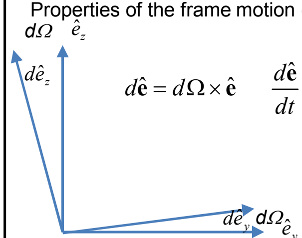
$$d\hat{e}_z = -d\Omega\hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Properties of the frame motion (rotation only):



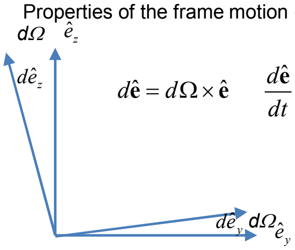
$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega) \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega \hat{\mathbf{e}}_y - d\Omega \hat{\mathbf{e}}_z = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

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Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

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Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial}$

Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

Coriolis force

Centrifugal force

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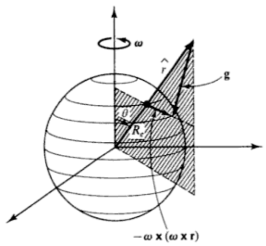
Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3\times10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Earth's gravity

Support force

Main contributions :

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

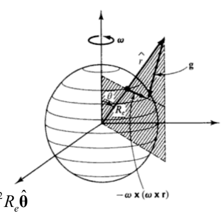
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



9.80 m/s²

0.03 m/s²

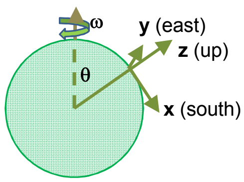
Foucault pendulum

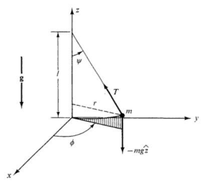
http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

Equation of motion on Earth's surface

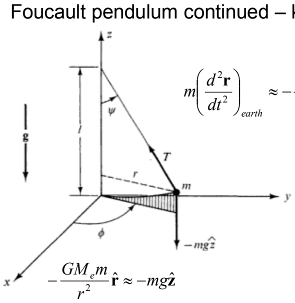
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + (\dot{x}\cos\theta + \dot{z}\sin\theta)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

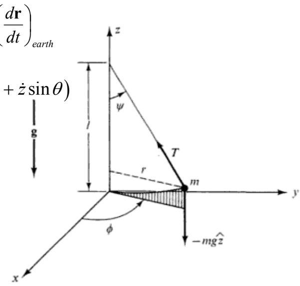
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\varphi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -T\sin\psi\sin\varphi - 2m\omega(\dot{x}\cos\theta + \dot{z}\sin\theta)$$

$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin\theta$$



Further approximation :

$\psi << 1; \quad \ddot{z} \approx 0; \quad T \approx mg$

$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$

$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$

Also note that :

$x \approx \ell\sin\psi\cos\phi$

$y \approx \ell\sin\psi\sin\phi$

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$
$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$

Try to find a solution of the form:

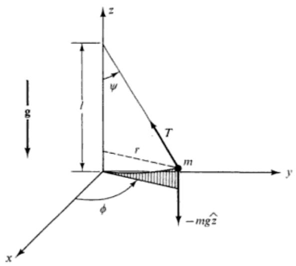
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non-trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\,2} + \frac{g}{\ell}}$$



Foucault pendulum continued – coupled equations:

Solution continued:

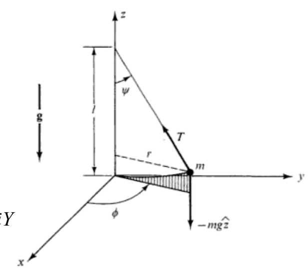
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non-trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\,2} + \frac{g}{\ell}}$$

Amplitude relationship: $X = iY$



General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$
$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$
$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\,2} + \frac{g}{\ell}} \approx \omega_\perp \pm \sqrt{\frac{g}{\ell}}$$

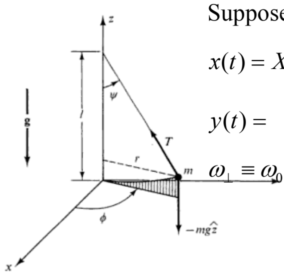
since $\omega_\perp \approx 7\times10^{-5}\cos\theta \; rad/s << \sqrt{\frac{g}{\ell}}$

Suppose: $x(0) = X_0 \quad y(0) = 0$

$$x(t) = X_0\cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos(\omega_\perp t)$$
$$y(t) = -X_0\cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin(\omega_\perp t)$$

Note that
$\omega = \frac{2\pi}{24\cdot 3600\text{ s}} = 7\times10^{-5}$ rad/sec

Summary of approximate solution for Foucault pendulum:

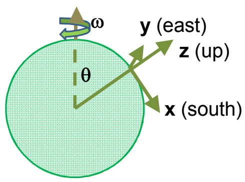


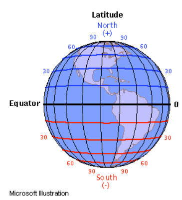
Suppose: $x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\sin\left(\omega_\perp t\right)$$

$$\omega_\perp \equiv \omega_0 \cos\theta$$





$$\omega_\perp \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\sin\left(\omega_\perp t\right)$$