PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 7:

Calculus of variation;
Continue reading Chapt. 3

1. Brachistochrone problem
2. Calculus of variation with constraints
3. Application to classical mechanics

PHY 711 Classical Mechanics and Mathematical Methods

MWF 9 AM-9:50 AM | OPL 107 | http://www.wfu.edu/~natalie/f17phy711/

Instructor: Natalie Holzwarth | Phone:758-5510 | Office:300 OPL | e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in an non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017	Chap. 3	Calculus of variations	#5	9/11/2017
7	Mon, 9/11/2017	Chap.3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017				
9	Fri, 9/15/2017				
10	Mon, 9/18/2017				

PHY 711 – Assignment #6

September 13, 2016

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$

   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$

   (c) Evaluate
   $$\frac{df}{dt}.$$

   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = e^{-t/\tau}.$$
   Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

Review: for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$ ⇦ Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

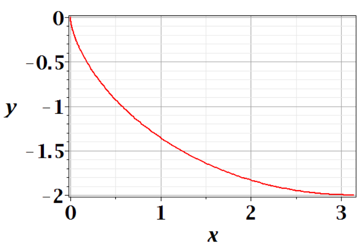
$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$ ⇦ Alternate Euler-Lagrange equation

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2*?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx$$ because $\frac{1}{2}mv^2 = -mgy$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

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$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a \qquad \text{Let} \quad y=-2a\sin^2\tfrac{\theta}{2}=a(\cos\theta-1)$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1} \qquad -\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx \qquad x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

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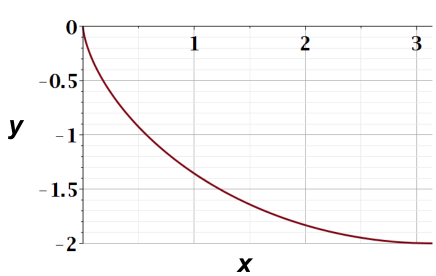
Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



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Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Euler-Lagrange equation:

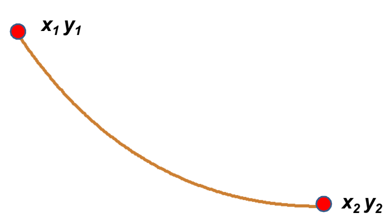
$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$(\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$(\rho g y + \lambda)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$y(x_2) = y_2$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics

$x \to t$ (time)

$y \to q$ (generalized coordinate)

$f \to L$ (Lagrangian)

$I \to A$ (action)

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

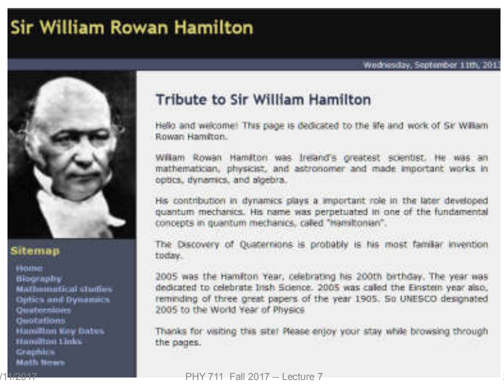
Application to particle dynamics

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g*.

$$m\frac{d^2y}{dt^2} = -mg$$

$$y(t) = y_i + v_i t - \tfrac{1}{2}gt^2$$

http://www.hamilton2005.ie/



Sir William Rowan Hamilton

Wednesday, September 11th, 2013

Tribute to Sir William Hamilton

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

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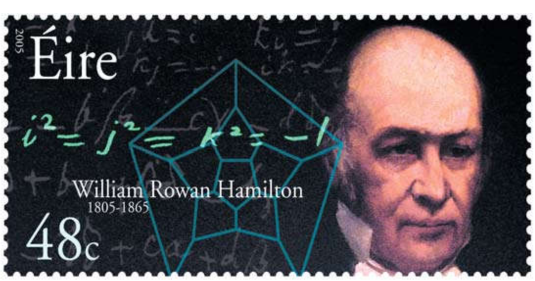

http://rjlipton.wordpress.com

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy$$

Hamilton's principle states:

$$S\equiv\int_{t_i}^{t_f}L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S\equiv\int_{t_i}^{t_f}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy\right)dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y}-\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}=0$$

$$\Rightarrow -mg-\frac{d}{dt}m\dot{y}=0$$

$$\Rightarrow \frac{d}{dt}\frac{dy}{dt}=-g \qquad y(t)=y_i+v_it-\tfrac{1}{2}gt^2$$

Check:

$$S\equiv\int_{t_i}^{t_f}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy\right)dt$$

Assume $t_i=0,\ y_i=h\equiv\frac{1}{2}gT^2;\quad t_f=T,\ y_f=0$

Trial trajectories:

$$y_1(t)=\tfrac{1}{2}gT^2\left(1-t/T\right)=h-\tfrac{1}{2}gTt$$

$$y_2(t)=\tfrac{1}{2}gT^2\left(1-t^2/T^2\right)=h-\tfrac{1}{2}gt^2$$

$$y_3(t)=\tfrac{1}{2}gT^2\left(1-t^3/T^3\right)=h-\tfrac{1}{2}gt^3/T$$

Maple says:

$$S_1=-0.125mg^2T^3$$

$$S_2=-0.167mg^2T^3$$

$$S_3=-0.150mg^2T^3$$