

**PHY 711 Classical Mechanics and
Mathematical Methods
9-9:50 AM MWF Olin 107**

Plan for Lecture 8:

Continue reading Chapter 3

- 1. D'Alembert's principle**
- 2. Hamilton's principle**
- 3. Lagrange's equation in
generalized coordinates**

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WFU Physics

People | Events and News | Undergraduate | Graduate | Research | Resources

Events

Colloquium: Sep. 13, 2017 at 4 PM
 WFU Physics Colloquium TITLE: "Machine Learning in Experimental Nuclear Physics"
 SPEAKER: Professor Michelle Kuchera
 Department of Physics Davidson College
 Davidson, NC TIME: Wed. Sep. 13, 2017 at 4:00 PM PLACE: ...

Career Event: Sept. 14 at 12:30 pm
 WFU Physics Career Advising Event TITLE:
 "The Business of Space: and other ways for
 physicists to use a degree from WFU"
 SPEAKER: Michael Heine CEO, Heine
 Ventures and Wake Forest Alum (1972) ...

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WFU Physics Colloquium

TITLE: "Machine Learning in Experimental Nuclear Physics"

SPEAKER: [Professor Michelle Kuchera](#)
 Department of Physics
 Davidson College
 Davidson, NC

TIME: Wed. Sep. 13, 2017 at 4:00 PM

PLACE: George P. Williams, Jr. Lecture Hall,
 (Olin 101)

ABSTRACT

The atomic nucleus was discovered by Ernest Rutherford in 1911. Over 100 years later, as the limits of nuclear existence are beginning to be explored, we are still working toward a complete understanding of the structure and properties of all nuclei. Advanced accelerators and detectors are used to study reactions involving rare nuclei. This talk will discuss the computational challenges and methods of "big data" detector systems.

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PHY 711 Classical Mechanics and Mathematical Methods

MWF 9 AM-9:50 AM | OPL 107 | <http://www.wfu.edu/~natalie/711phy711/>

Instructor: [Natalie Holzwarth](#) Phone: 758-5510 Office: 300 OPL e-mail: natalie@wfu.edu

Course schedule

(Preliminary schedule – subject to frequent adjustment.)

	Date	F&W	Reading Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/5/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in a non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017	Chap. 3	Calculus of variations	#5	9/11/2017
7	Mon, 9/11/2017	Chap. 3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017	Chap. 3	Lagrangian Mechanics	#7	9/15/2017
9	Fri, 9/15/2017				
10	Mon, 9/18/2017				

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PHY 711 – Assignment #7

September 12, 2017

1. Consider a Lagrangian describing the motion of a particle of mass m and charge q given by

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}B\dot{y}x.$$

Here c denotes the speed of light and B represents the magnitude of a constant magnetic field along the z -axis. Determine the Euler-Lagrange equations of motion for the particle and discuss how the motion compares with the similar example discussed in class.

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Jean d'Alembert 1717-1783
French mathematician and philosopher



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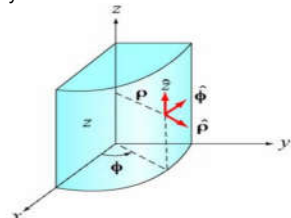
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Digression -- notion of generalized coordinates

Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$

Cylindrical coordinates



$$\begin{aligned}x &= \rho \cos \phi \\y &= \rho \sin \phi \\z &= z \\ \rho &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x) \\ z &= z\end{aligned}$$

Figure B.2.4 Cylindrical coordinates

(Figure taken from 8.02 handout from MIT.)

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Spherical coordinates

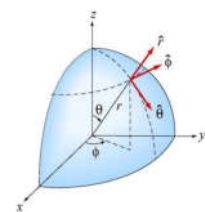


Figure B.3.1 Spherical coordinates

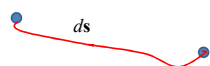
(Figure taken from 8.02 handout from MIT.)

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D'Alembert's principle:



Generalized coordinates :
 $q_\sigma(\{x_i\})$

Newton's laws :

$$\mathbf{F} - m\mathbf{a} = 0 \quad \Rightarrow \quad (\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0$$

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

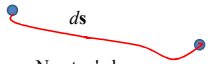
$$\text{For a conservative force: } F_i = -\frac{\partial U}{\partial x_i}$$

$$\mathbf{F} \cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

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Generalized coordinates :
 $q_\sigma(\{x_i\}) \quad x \leftrightarrow x_1$
 $y \leftrightarrow x_2$
 $z \leftrightarrow x_3$

Newton's laws :
 $\mathbf{F} \cdot d\mathbf{s} = 0 \Rightarrow (\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0$

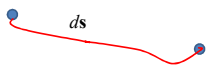
$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i m \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt} \left(m \dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

Claim : $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt} \left(\frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial q_\sigma} \right) \delta q_\sigma$$

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$x_i = x_i(\{q_\sigma(t)\}, t)$

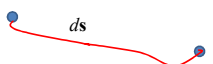
Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial x_i}{\partial t}$ Therefore: $\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \frac{\partial x_i}{\partial q_\sigma}$

Claim: $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\sum_\sigma \frac{\partial^2 x_i}{\partial q_\sigma \partial q_\sigma} \dot{q}_\sigma + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \sum_\sigma \frac{\partial^2 x_i}{\partial q_\sigma \partial q_\sigma} \dot{q}_\sigma + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$$

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Generalized coordinates :
 $q_\sigma(\{x_i\})$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt} \left(\frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial q_\sigma} \right) \delta q_\sigma$$

Define -- kinetic energy : $T \equiv \sum_i \frac{1}{2} m \dot{x}_i^2$

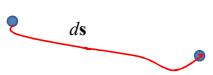
$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

Recall:

$$\mathbf{F} \cdot d\mathbf{s} = - \sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = - \sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = - \sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

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Generalized coordinates :
 $q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

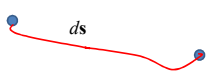
$$= -\sum_\sigma \left(\frac{d}{dt} \frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$= -\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$L(q_\sigma, \dot{q}_\sigma; t) = T - U$

Note : This is only true if $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

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Generalized coordinates :
 $q_\sigma(\{x_i\})$

Define-- Lagrangian: $L \equiv T - U$
 $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral: $S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

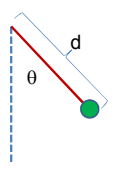
$\rightarrow$ Hamilton's principle

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Euler - Lagrange equations : $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta, \dot{\theta}) = \frac{1}{2} m d^2 \dot{\theta}^2 - m g (d - d \cos \theta)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} + m g d \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{d} \sin \theta$$

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Another example: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgd \cos \beta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt} (I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt} (I_1 \dot{\beta}) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt} (I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})) = 0$$

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Example – simple harmonic oscillator

$$T = \frac{1}{2} m \dot{x}^2$$

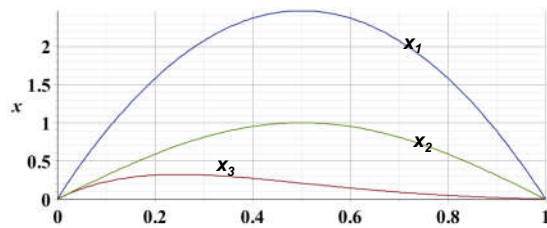
$$U = \frac{1}{2} m \omega^2 x^2$$

$$\text{Assume } x(0) = 0 \text{ and } x(\frac{\pi}{\omega}) = 0 \quad S = \int_0^{\pi/\omega} (T - U) dt$$

$$\text{Trial functions } x_1(t) = A \sin(\omega t) \quad S_1 = 0$$

$$x_2(t) = A \omega t \cdot (\pi - \omega t) \quad S_2 = 0.067 A^2 m \omega^2$$

$$x_3(t) = A e^{-\omega t} \sin(\omega t) \quad S_3 = 0.062 A^2 m \omega^2$$



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Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

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Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma} \quad \text{does not contribute to the result.}$$

$\Rightarrow$ How can we represent velocity -dependent forces?

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Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

$$\text{Lorentz force: } \mathbf{F} = q(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$$

$$x\text{-component: } F_x = q(E_x + \frac{1}{c} (\mathbf{v} \times \mathbf{B})_x)$$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x\text{-component: } \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) = 0$$

$$\text{Apparently: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$\text{Answer: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

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Lorentz forces, continued:

$$x\text{-component of Lorentz force: } F_x = q(E_x + \frac{1}{c} (\mathbf{v} \times \mathbf{B})_x)$$

$$\text{Suppose: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Consider: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x(\mathbf{r}, t)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \frac{dA_x(\mathbf{r}, t)}{dt} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

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Lorentz forces, continued:

$$\begin{aligned}
 -\frac{\partial U}{\partial x} &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right) \\
 \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} &= -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right) \\
 F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\
 &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\
 &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) \\
 &= qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x
 \end{aligned}$$

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Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

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Example Lorentz force

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Suppose } \mathbf{E}(\mathbf{r}, t) \equiv 0, \quad \mathbf{B}(\mathbf{r}, t) \equiv B_0 \hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2} B_0 (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad \frac{d}{dt} m\dot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + y\dot{x})$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + y\dot{x})$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained
from direct application of Newton's laws :

$$m\ddot{\mathbf{r}} = \frac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

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Example Lorentz force -- continued

Consider formulation with different Gauge : $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 x \dot{y}$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} (m\dot{y}) + \frac{q}{c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$

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Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \quad \dot{x}(t) = V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \quad \dot{y}(t) = V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{z} = 0 \quad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{mc}{qB_0} V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$y(t) = y_0 + \frac{mc}{qB_0} V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$z(t) = z_0 + V_{0z} t$$

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