PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 9:

Continue reading Chapter 3 & 6

1. Summary & review
2. Lagrange's equations with constraints

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

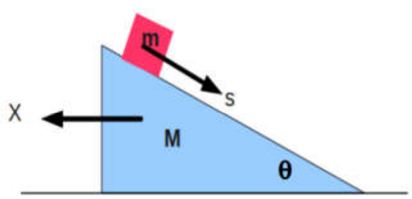
	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in an non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017	Chap. 3	Calculus of variations	#5	9/11/2017
7	Mon, 9/11/2017	Chap.3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017	Chap. 3	Lagrangian Mechanics	#7	9/15/2017
9	Fri, 9/15/2017	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/20/2017
10	Mon, 9/18/2017				
11	Wed, 9/20/2017				
12	Fri, 9/22/2017				
13	Mon, 9/25/2017				
14	Wed, 9/27/2017				
15	Fri, 9/29/2017				
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017				

PHY 711 - Assignment #8

PDF VERSION

9/15/2017

Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ.)

Comment on HW#7

Given $f(q(t),\dot{q}(t),t)$

$$\frac{df}{dt}=\frac{\partial f}{\partial q}\frac{dq}{dt}+\frac{\partial f}{\partial \dot{q}}\frac{d\dot{q}}{dt}+\frac{\partial f}{\partial t}$$

$$=\frac{\partial f}{\partial q}\dot{q}+\frac{\partial f}{\partial \dot{q}}\frac{d\dot{q}}{dt}+\frac{\partial f}{\partial t}$$

$$\frac{\partial}{\partial \dot{q}}\frac{df}{dt}=\frac{\partial^2 f}{\partial \dot{q}\partial q}\dot{q}+\frac{\partial f}{\partial q}+\frac{\partial^2 f}{\partial \dot{q}\partial \dot{q}}\frac{d\dot{q}}{dt}+\frac{\partial^2 f}{\partial \dot{q}\partial t}$$

Comments on generalized coordinates:

$$L=L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial q_\sigma}=0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

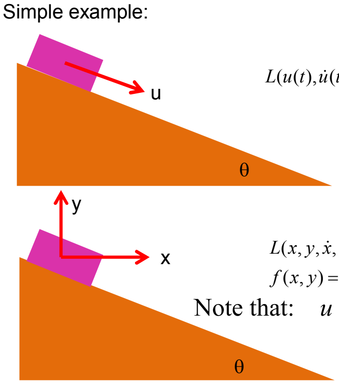
Lagrangian : $L=L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$

Constraints : $f_j=f_j(\{q_\sigma(t)\},t)=0$

Lagrange multipliers

Modified Euler - Lagrange equations : $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial q_\sigma}+\sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma}=0$

Simple example:



$$L(u(t),\dot{u}(t))=\tfrac{1}{2}m\dot{u}^2+mgu\sin\theta$$

$$L(x,y,\dot{x},\dot{y})=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)-mgy$$

$$f(x,y)=\sin\theta\, x+\cos\theta\, y=0$$

Note that: $u=x\cos\theta-y\sin\theta$

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\,\ddot{x} + \cos\theta\,\ddot{y} = 0$$

$$\Rightarrow \lambda = mg\cos\theta$$ ← Force of constraint; normal to incline

$$\left(\cos\theta\,\ddot{x} - \sin\theta\,\ddot{y}\right) = g\sin\theta$$

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)dt$$

$$\delta S = 0 = \int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma\right)dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\},t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma$$

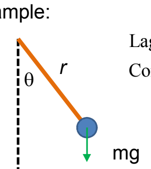
Euler-Lagrange equations with constraints:

Lagrangian: $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$

Constraints: $f_j = f_j\left(\{q_\sigma(t)\},t\right) = 0$

Modified Euler-Lagrange equations: $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Example:



Lagrangian: $L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$

Constraints: $f = r - \ell = 0$

Example continued:

Lagrangian : $L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+mgr\cos\theta$

Constraints : $f=r-\ell=0$

$$\frac{d}{dt}m\dot{r}-mr\dot{\theta}^2-mg\cos\theta+\lambda=0$$

$$\frac{d}{dt}mr^2\dot{\theta}+mgr\sin\theta=0$$

$$\dot{r}=0=\ddot{r} \qquad r=\ell$$

$$\Rightarrow\ \ddot{\theta}=-\frac{g}{\ell}\sin\theta$$

$$\Rightarrow\ \lambda=m\ell\dot{\theta}^2+mg\cos\theta$$

Another example:

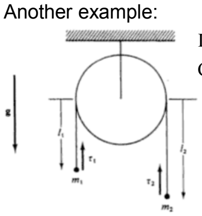


Figure 19.1 Atwood's machine.

Lagrangian : $L=\frac{1}{2}m_1\dot{\ell}_1^2+\frac{1}{2}m_2\dot{\ell}_2^2+m_1g\ell_1+m_2g\ell_2$

Constraints : $f=\ell_1+\ell_2-\ell=0$

$$\frac{d}{dt}m_1\dot{\ell}_1-m_1g+\lambda=0$$

$$\frac{d}{dt}m_2\dot{\ell}_2-m_2g+\lambda=0$$

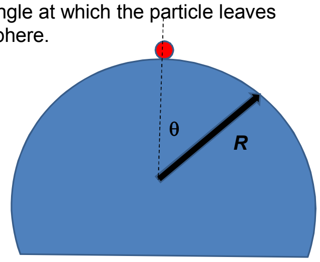
$$\dot{\ell}_1+\dot{\ell}_2=0=\ddot{\ell}_1+\ddot{\ell}_2$$

$$\Rightarrow\lambda=\frac{2m_1m_2}{m_1+m_2}g$$

$$\ddot{\ell}_1=-\ddot{\ell}_2=\frac{m_1-m_2}{m_1+m_2}g$$

Another example:

A particle of mass *m* starts at rest on top of a smooth fixed hemisphere of radius *R*. Find the angle at which the particle leaves the hemisphere.



Example continued

Constraint Equation: $f(r,\theta) = r - R = 0$

Lagrangian: $L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\left(1-\cos\theta\right)$

Euler - Lagrangian equations :

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \qquad mr\dot{\theta}^2 + mg\left(1-\cos\theta\right) - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Example continued

$$mr\dot{\theta}^2 + mg\left(1-\cos\theta\right) - m\ddot{r} + \lambda = 0$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Also note, from conservation of energy:

$$\frac{1}{2}mR^2\dot{\theta}^2 - mgR\left(1-\cos\theta\right) = 0$$

$$\dot{\theta}^2 = \frac{2g}{R}\left(1-\cos\theta\right)$$

Using constraint:

$$mR\dot{\theta}^2 + mg\left(1-\cos\theta\right) + \lambda = 0$$

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

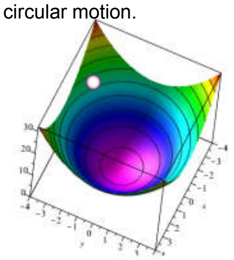
$$\ddot{\theta} = \frac{g}{R}\sin\theta \qquad \lambda = mg\cos\theta - mR\dot{\theta}^2 - mg$$

In this case λ, is not exactly the force of constraint F_C

$$F_C = mg\cos\theta - mR\dot{\theta}^2 = mg\left(3\cos\theta - 2\right) \quad \text{for } F_C \geq 0$$

$\Rightarrow$ for $\cos\theta = \frac{2}{3}$ or $\theta = 48.2$ deg, object falls from hemisphere

Consider a particle of mass *m* moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

Transform to polar coordinates;

$x = r\cos\phi \qquad y = r\sin\phi$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad 0 - \frac{d}{dt}mr^2\dot{\phi} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \Rightarrow \quad \text{Let } mr^2\dot{\phi} \equiv \ell_z \quad \text{(constant)}$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1 + 4c^2r^2\right)\right) = 0$$

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1 + 2c^2r^2\right)\right) = 0$$

Stable solution when these terms add to 0:

$$\ell_z = mr_0^2\sqrt{2gc} = mz_0\sqrt{\frac{2g}{c}}$$

Analysis of stable (circular) motion

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1 + 2c^2r^2\right)\right) = 0$$

$$\ell_z = mr_0^2\sqrt{2gc} = mz_0\sqrt{\frac{2g}{c}}$$

Let $r = r_0 + \delta r$ keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1 + 2c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1 + 2c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1 + 2c^2r_0^2}}t + \alpha\right)$$