PHY 741 Quantum Mechanics
12-12:50 PM MWF Olin 103

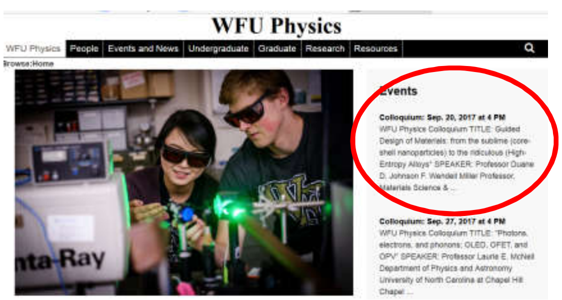
Plan for Lecture 11:
Read Chapter # 9 in Shankar
Commutator formalism and
the Heisenburg Uncertainty Relations

1. **Commutator definitions and relationships**
2. **General theorem concerning Heisenburg uncertainty relations**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Review of basic principles	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Linear vector spaces	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Linear vector spaces	#3	9/6/2017
4	Mon, 9/04/2017	Chap. 4	Principles of Quantum Mechanics	#4	9/8/2017
5	Wed, 9/06/2017	Chap. 5	Examples in 1 dimension		
6	Fri, 9/08/2017	Chap. 5	Schrödinger equation in one-dimension	#5	9/13/2017
7	Mon, 9/11/2017	Chap. 5	Schrödinger equation in one-dimension		
8	Wed, 9/13/2017	Chap. 7	Schrödinger equation in one-dimension	#6	9/15/2017
9	Fri, 9/15/2017	Chap. 7	Schrödinger equation in one-dimension	#7	9/20/2017
10	Mon, 9/18/2017	Chap. 5 and 7	Schrödinger equation in one-dimension		
11	Wed, 9/20/2017	Chap. 9	Commutator formalism	#8	9/22/2017
12	Fri, 9/22/2017				
13	Mon, 9/25/2017				
14	Wed, 9/27/2017				
15	Fri, 9/29/2017				
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		



WFU Physics Colloquium

TITLE: Guided Design of Materials: from the sublime (core-shell nanoparticles) to the ridiculous (High-Entropy Alloys*

SPEAKER: Professor Duane D. Johnson
F. Wendell Miller Professor,
Materials Science & Engineering, Iowa State University
Chief Scientist, Ames Laboratory/U.S. DOE, Ames, Iowa

TIME: Wed. Sep. 20, 2017 at 4:00 PM

PLACE: George P. Williams, Jr. Lecture Hall, (Olin 101)

ABSTRACT

Since the Iron Age, complex alloying effects have yielded desirable and unusual behaviors whose origins were difficult to unravel. Here we address intriguing behavior from alloying that occur in metallic nanoparticles to bulk complex solid solutions.

Commutator formalism in quantum mechanics

Definition:
Given two Hermitian operators A and B, their commutator is
$[A,B] \equiv AB - BA$

Theorem:
Given Hermitian operators A, B, C such that
$[A,B] = iC$,

it follows that $\Delta A \Delta B \geq \frac{1}{2}\left|\langle C \rangle\right|$

Note that:

$$[A,B]^\dagger = \left(iC\right)^\dagger$$

$$\left(AB - BA\right)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -iC^\dagger$$

$$= BA - AB = -iC$$

Calculation of the variance:

$$(\Delta A)^2 \equiv \langle\psi|\left(A - \langle A\rangle\right)^2|\psi\rangle$$
$$= \left\langle\left(A - \langle A\rangle\right)\psi\middle|\left(A - \langle A\rangle\right)\psi\right\rangle$$

Similarly,

$$(\Delta B)^2 \equiv \langle\psi|\left(B - \langle B\rangle\right)^2|\psi\rangle$$
$$= \left\langle\left(B - \langle B\rangle\right)\psi\middle|\left(B - \langle B\rangle\right)\psi\right\rangle$$

Define $|\psi_A\rangle \equiv \left|\left(A - \langle A\rangle\right)\psi\right\rangle$

$|\psi_B\rangle \equiv \left|\left(B - \langle B\rangle\right)\psi\right\rangle$

Schwarz inequality:

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq \left|\langle\psi_A|\psi_B\rangle\right|^2$$

Define $|\psi_A\rangle \equiv \left|(A-\langle A\rangle)\psi\right\rangle$ and $|\psi_B\rangle \equiv \left|(B-\langle B\rangle)\psi\right\rangle$

Schwarz inequality:

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq \left|\langle\psi_A|\psi_B\rangle\right|^2$$

$$\langle\psi_A|\psi_B\rangle = \langle\psi|(A-\langle A\rangle)(B-\langle B\rangle)|\psi\rangle$$

$$(A-\langle A\rangle)(B-\langle B\rangle) = \frac{1}{2}\big((A-\langle A\rangle)(B-\langle B\rangle)+(B-\langle B\rangle)(A-\langle A\rangle)\big)$$

$$+\frac{1}{2}\big((A-\langle A\rangle)(B-\langle B\rangle)-(B-\langle B\rangle)(A-\langle A\rangle)\big)$$

$$= F + \frac{i}{2}C$$

$$\langle\psi_A|\psi_B\rangle = \langle\psi|(A-\langle A\rangle)(B-\langle B\rangle)|\psi\rangle$$

$$= \langle\psi|F|\psi\rangle + \frac{i}{2}\langle\psi|C|\psi\rangle$$

$$\left|\langle\psi_A|\psi_B\rangle\right|^2 = \left|\langle\psi|F|\psi\rangle\right|^2 + \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2$$

Putting it all together:

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq \left|\langle\psi_A|\psi_B\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2$$

$$\Rightarrow (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left|\langle C\rangle\right|^2$$

$[A,B] = iC \qquad \Delta A\Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|$

Example: $A = X, \quad B = P$

$[X,P] = i\hbar \qquad \Delta X\Delta P \geq \frac{\hbar}{2}$

Other uses of the commutator

Consider a wavefunction that solves the time-dependent Schroedinger equation:

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi$$

Expectation value of operator F:

$$\langle F\rangle=\langle\psi|F|\psi\rangle$$

Time derivative of $\langle F\rangle$:

$$\frac{d\langle F\rangle}{dt}=\left\langle\frac{\partial\psi}{\partial t}\right|F|\psi\rangle+\langle\psi|\frac{\partial F}{\partial t}|\psi\rangle+\langle\psi|F\left|\frac{\partial\psi}{\partial t}\right\rangle$$

$$=\frac{i}{\hbar}\langle H\psi|F|\psi\rangle+\langle\psi|\frac{\partial F}{\partial t}|\psi\rangle-\frac{i}{\hbar}\langle\psi|F|H\psi\rangle$$

$$=\frac{1}{i\hbar}\langle\psi|[F,H]|\psi\rangle+\langle\psi|\frac{\partial F}{\partial t}|\psi\rangle=\langle\psi|\left(\frac{1}{i\hbar}[F,H]+\frac{\partial F}{\partial t}\right)|\psi\rangle$$

Some other useful commutator relationships:

For $[A,[A,B]]=0$ and $[B,[A,B]]=0$

$$[A,B^n]=nB^{n-1}[A,B]$$

$$[A^n,B]=nA^{n-1}[A,B]$$

Evaluation of the commutator

$$[A,B]|\psi\rangle=AB|\psi\rangle-BA|\psi\rangle$$

Example:

$$[f(X),P]|\psi\rangle=f(X)P|\psi\rangle-Pf(X)|\psi\rangle=i\hbar\frac{\partial f}{\partial X}|\psi\rangle$$

Note that: $P=-i\hbar\frac{\partial}{\partial X}$

Angular momentum in three dimensions

$$L_x=YP_z-ZP_y$$

$$L_y=ZP_x-XP_z$$

$$L_z=XP_y-YP_x$$

Consider:

$$[L_x,Y]=[YP_z-ZP_y,Y]=-Z[P_y,Y]=i\hbar Z$$

$$[L_x,P_y]=[YP_z-ZP_y,P_y]=P_z[Y,P_y]=i\hbar P_z$$

$$[L_x,L_y]=[YP_z-ZP_y,ZP_x-XP_z]=i\hbar L_z$$

Summary:

$$[L_x,L_y]=i\hbar L_z \qquad [L_y,L_z]=i\hbar L_x \qquad [L_z,L_x]=i\hbar L_y$$

Summary:

$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$

Consider $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$\left[\mathbf{L}^2, L_z\right] = \left[L_x^2 + L_y^2, L_z\right]$$
$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$
$$= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y$$
$$= 0$$

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Commutator relations for multiple particles

Consider N particles moving in one-dimension

$\{X_1, X_2 \ldots X_N, P_1, P_2 \ldots P_N\}$

$$[X_i, P_j] = i\hbar\delta_{ij}$$
$$[X_i, X_j] = 0$$
$$[P_i, P_j] = 0$$

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