PHY 741 Quantum Mechanics
12-12:50 PM MWF Olin 103

Plan for Lecture 13:
Reading in Shankar: Finish Chapter 10
Skim Chapter 11
Start Chapter 12

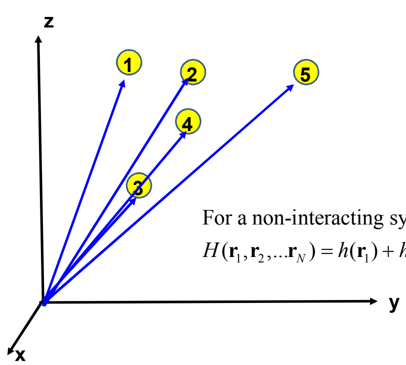
1. **Multiparticle systems**
   a. **Distinguishable, Fermi, Bose**
   b. **Second quantized formalism**
2. **Angular momentum**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Review of basic principles	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Linear vector spaces	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Linear vector spaces	#3	9/6/2017
4	Mon, 9/04/2017	Chap. 4	Principles of Quantum Mechanics	#4	9/8/2017
5	Wed, 9/06/2017	Chap. 5	Examples in 1 dimension		
6	Fri, 9/08/2017	Chap. 5	Schrödinger equation in one-dimension	#5	9/13/2017
7	Mon, 9/11/2017	Chap. 5	Schrödinger equation in one-dimension		
8	Wed, 9/13/2017	Chap. 7	Schrödinger equation in one-dimension	#6	9/15/2017
9	Fri, 9/15/2017	Chap. 7	Schrödinger equation in one-dimension	#7	9/20/2017
10	Mon, 9/18/2017	Chap. 5 and 7	Schrödinger equation in one-dimension		
11	Wed, 9/20/2017	Chap. 9	Commutator formalism	#8	9/22/2017
12	Fri, 9/22/2017	Chap. 10	Quantum mechanics of multiparticle systems	#9	9/25/2017
13	Mon, 9/25/2017	Chap. 10-12	Multiparticle systems and angular momentum		
14	Wed, 9/27/2017	Chap. 12	Eigenstates of angular momentum		
15	Fri, 9/29/2017	Chap. 1,4,5,7,9,10,12	Review		
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017				

Quantum mechanical treatment of multiparticle systems



For a non-interacting system:

$$H(\mathbf{r}_1,\mathbf{r}_2,\ldots\mathbf{r}_N) = h(\mathbf{r}_1) + h(\mathbf{r}_2) + \ldots h(\mathbf{r}_N)$$

Quantum mechanical treatment of multiparticle systems

For a non-interacting system:

$$H(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = h(\mathbf{r}_1) + h(\mathbf{r}_2) + ...h(\mathbf{r}_N)$$

Energy eigenstates:

$$H(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N)\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = E\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N)$$

Simplification for separable Hamiltonian

For: $h(\mathbf{r}_1)\varphi_a(\mathbf{r}_1) = \varepsilon_a\varphi_a(\mathbf{r}_1)$

$h(\mathbf{r}_2)\varphi_b(\mathbf{r}_2) = \varepsilon_b\varphi_b(\mathbf{r}_2)$

Solution to the many particle problem

$$\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = \varphi_a(\mathbf{r}_1)\varphi_b(\mathbf{r}_2)...\varphi_z(\mathbf{r}_N)$$

$$E = \varepsilon_a + \varepsilon_b + ...\varepsilon_z$$

Refinement of the results for treatment of distinguisable or indistinguishable particles

For distinguishable particles:

$$\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = \varphi_a(\mathbf{r}_1)\varphi_b(\mathbf{r}_2)...\varphi_z(\mathbf{r}_N)$$

Two types of indistinguishable particles:

Fermi particles: $\psi_F(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_i...\mathbf{r}_j...\mathbf{r}_N) = -\psi_F(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_j...\mathbf{r}_i...\mathbf{r}_N)$

$$\Rightarrow \psi_F(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_i...\mathbf{r}_j...\mathbf{r}_N) = \frac{1}{\sqrt{N!}}\sum_{\mathcal{P}}(-1)^{\mathcal{P}}\mathcal{P}\left(\varphi_a(\mathbf{r}_1)\varphi_b(\mathbf{r}_2)\varphi_c(\mathbf{r}_3)...\varphi_z(\mathbf{r}_N)\right)$$

Bose particles: $\psi_B(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_i...\mathbf{r}_j...\mathbf{r}_N) = +\psi_B(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_j...\mathbf{r}_i...\mathbf{r}_N)$

$$\Rightarrow \psi_B(\mathbf{r}_1,\mathbf{r}_2,..\mathbf{r}_i...\mathbf{r}_j...\mathbf{r}_N) = \frac{1}{\sqrt{N!}}\sum_{\mathcal{P}}\mathcal{P}\left(\varphi_a(\mathbf{r}_1)\varphi_b(\mathbf{r}_2)\varphi_c(\mathbf{r}_3)...\varphi_z(\mathbf{r}_N)\right)$$

Energy eigenstates: $H(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N)\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = E\psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N)$

$$E = \varepsilon_a + \varepsilon_b + ...\varepsilon_z$$

Treating multiparticle systems using "second" quantization formalism

Consider a non-interacting system:

$$H(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = h(\mathbf{r}_1) + h(\mathbf{r}_2) + ...h(\mathbf{r}_N)$$

For a system of non-interacting identical particles,

the single particle Hamiltonians $h(\mathbf{r}_1)$ are also identical.

Eigenstates of the single particle Hamiltonian:

$$h(\mathbf{r})\varphi_a(\mathbf{r}) = \varepsilon_a\varphi_a(\mathbf{r})$$

$$h(\mathbf{r})\varphi_b(\mathbf{r}) = \varepsilon_b\varphi_b(\mathbf{r})$$

$$\vdots$$

$$h(\mathbf{r})\varphi_z(\mathbf{r}) = \varepsilon_z\varphi_z(\mathbf{r})$$

We now assume that the single particle eigenstates $\{\varphi_a(\mathbf{r})\}$ span the function space available to each particle.

Treating multiparticle systems using "second" quantization formalism -- continued

$$h(\mathbf{r}_1) = \sum_\alpha |\varphi_\alpha(\mathbf{r}_1)\rangle \varepsilon_\alpha \langle \varphi_\alpha(\mathbf{r}_1)|$$

$$\langle \varphi_\alpha(\mathbf{r}_1) | \psi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)\rangle = n_\alpha$$

=number of times basis function $\varphi_\alpha(\mathbf{r}_1)$ appears in the product representation

Second quantization representation:

$$|\psi(\mathbf{r}_1, \mathbf{r}_2....\mathbf{r}_N)\rangle \Rightarrow |n_a n_b n_c....n_z\rangle$$

$$H(\mathbf{r}_1, \mathbf{r}_2....\mathbf{r}_N) \Rightarrow \sum_\alpha \varepsilon_\alpha N_\alpha$$

where the number operator acts as follows:

$$N_\alpha |n_a n_b n_c....n_z\rangle = n_\alpha |n_a n_b n_c....n_z\rangle$$

In general, the number operator can be expressed in terms of a product of two operators. For the case of Bose particles, these operators are very similar to the raising and lowering operators of the harmonic oscillator.

$$N_\alpha = b_\alpha^\dagger b_\alpha$$

Bose particle commutation relations:

$$[b_\alpha, b_\beta] \equiv b_\alpha b_\beta - b_\beta b_\alpha = 0$$

$$[b_\alpha^\dagger, b_\beta^\dagger] = 0$$

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}$$

Second quantization for Bose particles, continued

$$b_\alpha^\dagger b_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle$$

$$b_\alpha |n_\alpha\rangle = \sqrt{n_\alpha}\, |n_\alpha - 1\rangle$$

$$b_\alpha^\dagger |n_\alpha\rangle = \sqrt{n_\alpha + 1}\, |n_\alpha + 1\rangle$$

For example: $b_\alpha^\dagger |0_\alpha\rangle = |1_\alpha\rangle$

$$b_\alpha^\dagger |1_\alpha\rangle = \sqrt{2}\, |2_\alpha\rangle$$

$$\left(b_\alpha^\dagger\right)^n |0_\alpha\rangle = \sqrt{n!}\, |n_\alpha\rangle$$

$$\Rightarrow n_\alpha = 0, 1, 2, ...\infty$$

To represent 3 particles: $|n_1 n_2 n_3\rangle = \frac{\left(b_3^\dagger\right)^{n_3}}{\sqrt{n_3!}} \frac{\left(b_2^\dagger\right)^{n_2}}{\sqrt{n_2!}} \frac{\left(b_1^\dagger\right)^{n_1}}{\sqrt{n_1!}} |0\rangle$

Second quantization for Fermi particles

$$N_\alpha = f_\alpha^\dagger f_\alpha$$

Fermi particle anticommutation relations:

$$\left\{ f_\alpha, f_\beta \right\} \equiv f_\alpha f_\beta + f_\beta f_\alpha = 0$$

$$\left\{ f_\alpha^\dagger, f_\beta^\dagger \right\} = 0$$

$$\left\{ f_\alpha f_\beta^\dagger \right\} = \delta_{\alpha\beta}$$

Second quantized creation and annhilation Fermi operators

$$f_\alpha^\dagger f_\alpha \left| n_\alpha \right\rangle = n_\alpha \left| n_\alpha \right\rangle$$

$$f_\alpha \left| n_\alpha \right\rangle = \sqrt{n_\alpha} \left| 1 - n_\alpha \right\rangle$$

$$f^\dagger{}_\alpha \left| n_\alpha \right\rangle = \sqrt{1 - n_\alpha} \left| 1 - n_\alpha \right\rangle$$

Non-trivial operations:

$$f_\alpha \left| 0_\alpha \right\rangle = 0 \qquad f_\alpha \left| 1_\alpha \right\rangle = \left| 0_\alpha \right\rangle$$

$$f_\alpha^\dagger \left| 0_\alpha \right\rangle = \left| 1_\alpha \right\rangle \qquad f_\alpha^\dagger \left| 1_\alpha \right\rangle = 0$$

$$\Rightarrow n_\alpha = 0 \text{ or } 1$$

To represent 3 particles: $\left| n_1 n_2 n_3 \right\rangle = \left(f_3^\dagger \right)^{n_3} \left(f_2^\dagger \right)^{n_2} \left(f_1^\dagger \right)^{n_1} \left| 0 \right\rangle$

More general treatment of multiparticle system

$$H(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) = \sum_{i=1}^{N} h(\mathbf{r}_i) + V(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N)$$

interparticle interaction

Often: $V(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) = \sum_{i=1}^{N} \sum_{(i>j)\; j=1}^{N} v(\mathbf{r}_i - \mathbf{r}_j)$

In this case, the second quantized forms can be written

$$H(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_N) \Rightarrow \sum_\alpha \varepsilon_\alpha b_\alpha^\dagger b_\alpha + \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\beta^\dagger b_\gamma b_\delta \quad \text{for Bose particles}$$

$$H(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_N) \Rightarrow \sum_\alpha \varepsilon_\alpha f_\alpha^\dagger f_\alpha + \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} f_\alpha^\dagger f_\beta^\dagger f_\gamma f_\delta \quad \text{for Fermi particles}$$

Here $v_{\alpha\beta\gamma\delta}$ denotes matrix elements such as

$$v_{\alpha\beta\gamma\delta} = \int d^3 r_1 \int d^3 r_2 \; \varphi_\alpha^*(\mathbf{r}_1) \varphi_\beta^*(\mathbf{r}_2) v(\mathbf{r}_1 - \mathbf{r}_2) \varphi_\gamma(\mathbf{r}_1) \varphi_\delta(\mathbf{r}_2)$$

Back to single particle treatments; 1-3 dimensional space

Consider linear momentum in the x direction:

$$P_x = -i\hbar \frac{\partial}{\partial x}$$

$$P_x \left|\psi(x)\right\rangle = -i\hbar \left| \frac{\partial \psi(x)}{\partial x} \right\rangle$$

Note that $\psi(x+u) = \psi(x) + u\frac{\partial \psi(x)}{\partial x} + \frac{u^2}{2!}\frac{\partial^2 \psi(x)}{\partial x^2} + \ldots$

$$\psi(x+u) = \psi(x) + u\left(\frac{i}{\hbar}P_x\right)\psi(x) + \frac{u^2}{2!}\left(\frac{i}{\hbar}P_x\right)^2 \psi(x) + \ldots$$

Formally: $\psi(x+u) = e^{u\left(\frac{i}{\hbar}P_x\right)}\psi(x)$

$\Rightarrow$ The operator $e^{iuP_x/\hbar}$ is the "generator" of linear translations

Now consider angular momentum

Angular momentum in three dimensions

$$L_x = YP_z - ZP_y$$
$$L_y = ZP_x - XP_z$$
$$L_z = XP_y - YP_x$$

Consider:

$$[L_x, Y] = [YP_z - ZP_y, Y] = -Z[P_y, Y] = i\hbar Z$$
$$[L_x, P_y] = [YP_z - ZP_y, P_y] = P_z[Y, P_y] = i\hbar P_z$$
$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z] = i\hbar L_z$$

Summary:

$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$

Summary:

$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$

Consider $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$\left[\mathbf{L}^2, L_z\right] = \left[L_x^2 + L_y^2, L_z\right]$$
$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$
$$= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y$$
$$= 0$$

Consider L_z in spherical polar coordinates:

$$L_z = XP_y - YP_x = -i\hbar \frac{\partial}{\partial \varphi}$$

Following the same logic that we used for linear momentum

$$e^{i\alpha L_z/\hbar}\psi(x,\theta,\varphi) = \psi(x,\theta,\varphi+\alpha)$$

generator for rotations

Need to find eigenvalues and eigenfunctions of
L^2 and L_z.