PHY 741 Quantum Mechanics
12-12:50 AM MWF Olin 103

Plan for Lecture 32:

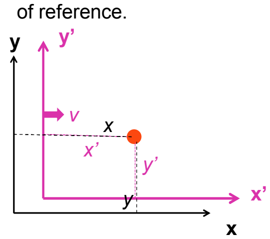
Chap. 20 in Shankar: The Dirac equation

1. **Some simple concepts of special theory of relativity**
2. **Energy and momentum relationships**
3. **Dirac equation for a free particle**

22	Mon, 10/23/2017	Chap. 15	Multi-electron atoms	#13	10/25/2017
23	Wed, 10/25/2017	Chap. 15	Multi-electron atoms	#14	10/30/2017
24	Fri, 10/27/2017		Effects of nuclear motion		
25	Mon, 10/30/2017	Chap. 17	Time-independent perturbation theory	#15	11/3/2017
26	Wed, 11/01/2017	Chap. 17	Time-independent perturbation theory		
27	Fri, 11/03/2017		Effects of a static magnetic field		
28	Mon, 11/06/2017	Chap. 18	Time-dependent perturbation theory	#16	11/10/2017
29	Wed, 11/08/2017	Chap. 18	Time-dependent perturbation theory		
30	Fri, 11/10/2017	Chap. 19	Scattering theory	#17	11/15/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 19	Scattering theory		
32	Fri, 11/17/2017	Chap. 20	Dirac Equation		
33	Mon, 11/20/2017	Chap. 20	Dirac Equation		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 20	Dirac Equation		
35	Wed, 11/29/2017	Chap. 20	Dirac Equation		
36	Fri, 12/01/2017	Chap. 1-20	Review		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum c is the same in all frames of reference.

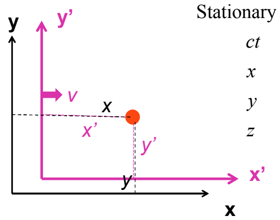


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note : $\omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note : $E^2 - p^2c^2 = E'^2 - p'^2 c^2$

Lorentz transformation of the velocity

Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

For an infinitesimal increment:

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

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Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Define: $u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$

$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x+v}{1+vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1+vu'_x/c^2)} = u_y$$

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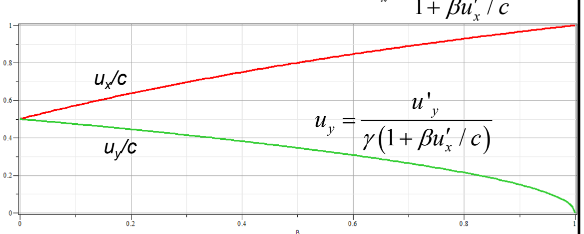
Example of velocity variation with β: $u_x = \frac{u'_x+v}{1+\beta u'_x/c}$



$$u_y = \frac{u'_y}{\gamma(1+\beta u'_x/c)}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

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Velocity transformations continued:

Consider: $u_x = \dfrac{u'_x + v}{1 + vu'_x/c^2} \quad u_y = \dfrac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \quad u_z = \dfrac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}} = \dfrac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'}\left(1 + vu'_x/c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v\left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$

Velocity 4-vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_u \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$

Some details:

$\gamma_u = \gamma_v \gamma_{u'}\left(1 + vu'_x/c^2\right) \quad \Rightarrow \left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{u'^2}{c^2}\right) = \left(1 - \dfrac{u^2}{c^2}\right)\left(1 + \dfrac{u_x v}{c^2}\right)^2$

where $u_x = \dfrac{u'_x + v}{1 + vu'_x/c^2} \quad u_y = \dfrac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \quad u_z = \dfrac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

$\left(\dfrac{u_x^2}{c^2} + \dfrac{u_y^2}{c^2} + \dfrac{u_z^2}{c^2}\right)\left(1 + \dfrac{u_x v}{c^2}\right)^2 = \left(\dfrac{u'_x}{c} + \dfrac{v}{c}\right)^2 + \left(\dfrac{u'^2_y}{c^2} + \dfrac{u'^2_z}{c^2}\right)\left(1 - \dfrac{v^2}{c^2}\right)$

$\dfrac{u^2}{c^2}\left(1 + \dfrac{u_x v}{c^2}\right)^2 = \dfrac{u'^2}{c^2}\left(1 - \dfrac{v^2}{c^2}\right) + \left(1 + \dfrac{u_x v}{c^2}\right)^2 - \left(1 - \dfrac{v^2}{c^2}\right)$

$\Rightarrow \left(1 - \dfrac{u^2}{c^2}\right)\left(1 + \dfrac{u_x v}{c^2}\right)^2 = \left(1 - \dfrac{u'^2}{c^2}\right)\left(1 - \dfrac{v^2}{c^2}\right)$

Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2c^2$$

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u^2}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0$, $E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1$, $E_K = mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check: } \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u^2\beta_u^2 + 1} = \gamma_u mc^2$$

Example: for an electron $mc^2 = 0.5\,\text{MeV}$

for $E = 200\,\text{GeV}$

$$\gamma_u = \frac{E}{mc^2} = 4\times10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u^2}} \approx 1 - \frac{1}{2\gamma_u^2} \approx 1 - 3\times10^{-12}$$

How do these relationships effect quantum mechanics?

Focusing on treatment of Fermi particles

Non-relativistic mechanics

$$E = \frac{\mathbf{p}^2}{2m}$$

$$\Downarrow$$

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

Relativistic mechanics

$$E^2 - \mathbf{p}^2c^2 = \left(mc^2\right)^2$$

$\Downarrow$ (with some license)

$$\left(E - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(E + \mathbf{p}\cdot\boldsymbol{\sigma}c\right) = \left(mc^2\right)^2$$

$$\Downarrow$$

$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi = \left(mc^2\right)^2\Psi$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi=\left(mc^2\right)^2\Psi$$

Let $\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi\equiv mc^2\Psi^R \qquad \Psi\equiv\Psi^L$

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Dirac's rearrangement: $\varphi^U=\Psi^R+\Psi^L$

$\varphi^L=\Psi^R-\Psi^L$

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=mc^2\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

Relativistic relationships – continued

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=mc^2\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

Further rearrangements:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

$$\Downarrow$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

Four component wavefunction of free Fermi particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c\\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

Assume $\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}\chi^U(\mathbf{k})\\ \chi^L(\mathbf{k})\end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{r}-iEt/\hbar}$

$$\Rightarrow \chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})$$

$$\chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

$$E^2=\hbar^2c^2\mathbf{k}^2+m^2c^4$$

$$E=\pm\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$$

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Pauli matrices: $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma}=\begin{pmatrix}k_z & k_x-ik_y\\ k_x+ik_y & -k_z\end{pmatrix}$$

$$\chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})\qquad \chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

Positive energy solutions: $E=\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}\qquad \chi^L_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\ \kappa_+\end{pmatrix}\qquad \kappa_z\equiv\frac{\hbar k_z c}{E+mc^2}$$

$$\chi^U_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}\qquad \chi^L_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\ -\kappa_z\end{pmatrix}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E+mc^2}$$

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Pauli matrices: $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma}=\begin{pmatrix}k_z & k_x-ik_y\\ k_x+ik_y & -k_z\end{pmatrix}$$

$$\chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})\qquad \chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

Negative energy solutions: $E=-\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\ \kappa_+\end{pmatrix}\qquad \chi^L_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}\qquad \kappa_z\equiv\frac{\hbar k_z c}{E-mc^2}$$

$$\chi^U_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\ -\kappa_z\end{pmatrix}\qquad \chi^L_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E-mc^2}$$

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What does this all mean?

Positive energy solutions: $E = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi_\uparrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix}1\\0\end{pmatrix} \qquad \chi_\uparrow^L(\mathbf{k}) = \mathcal{N}\begin{pmatrix}\kappa_z\\\kappa_+\end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi_\downarrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix}0\\1\end{pmatrix} \qquad \chi_\downarrow^L(\mathbf{k}) = \mathcal{N}\begin{pmatrix}\kappa_-\\-\kappa_z\end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

For $\hbar c|\mathbf{k}| \ll mc^2 \quad E \approx mc^2 + \frac{\hbar^2|\mathbf{k}|^2}{2m}$

$$\chi_\uparrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix}1\\0\end{pmatrix} \qquad \chi_\uparrow^L(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix}0\\0\end{pmatrix}$$

$$\chi_\downarrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix}0\\1\end{pmatrix} \qquad \chi_\downarrow^L(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix}0\\0\end{pmatrix}$$