PHY 741 Quantum Mechanics
12-12:50 AM MWF Olin 103

Plan for Lecture 33:

Chap. 20 in Shankar: The Dirac equation

1. **Dirac equation for a free particle**
2. **Dirac equation for a hydrogen atom**

22	Mon, 10/23/2017	Chap. 15	Multi-electron atoms	#13	10/25/2017
23	Wed, 10/25/2017	Chap. 15	Multi-electron atoms	#14	10/30/2017
24	Fri, 10/27/2017		Effects of nuclear motion		
25	Mon, 10/30/2017	Chap. 17	Time-independent perturbation theory	#15	11/3/2017
26	Wed, 11/01/2017	Chap. 17	Time-independent perturbation theory		
27	Fri, 11/03/2017		Effects of a static magnetic field		
28	Mon, 11/06/2017	Chap. 18	Time-dependent perturbation theory	#16	11/10/2017
29	Wed, 11/08/2017	Chap. 18	Time-dependent perturbation theory		
30	Fri, 11/10/2017	Chap. 19	Scattering theory	#17	11/15/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 19	Scattering theory		
32	Fri, 11/17/2017	Chap. 20	Dirac Equation		
33	Mon, 11/20/2017	Chap. 20	Dirac Equation		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 20	Dirac Equation		
35	Wed, 11/29/2017	Chap. 20	Dirac Equation		
36	Fri, 12/01/2017	Chap. 1-20	Review		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

How do these relationships effect quantum mechanics?
Focusing on treatment of Fermi particles

Non-relativistic mechanics

$$E = \frac{\mathbf{p}^2}{2m}$$
$$\Downarrow$$
$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

Relativistic mechanics

$$E^2 - \mathbf{p}^2c^2 = \left(mc^2\right)^2$$
$$\Downarrow \quad \text{(with some license)}$$
$$\left(E - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(E + \mathbf{p}\cdot\boldsymbol{\sigma}c\right) = \left(mc^2\right)^2$$
$$\Downarrow$$
$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi = \left(mc^2\right)^2\Psi$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi=\left(mc^2\right)^2\Psi$$

Let $\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi\equiv mc^2\Psi^R \qquad \Psi\equiv\Psi^L$

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Dirac's rearrangement: $\varphi^U=\Psi^R+\Psi^L$

$\varphi^L=\Psi^R-\Psi^L$

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=mc^2\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

Relativistic relationships – continued

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=mc^2\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

Further rearrangements:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

$$\Downarrow$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

Four component wavefunction of free Fermi particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c\\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

$$\text{Assume}\quad \begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}\chi^U(\mathbf{k})\\ \chi^L(\mathbf{k})\end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{r}-iEt/\hbar}$$

$$\Rightarrow \chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})$$

$$\chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

$$E^2=\hbar^2c^2\mathbf{k}^2+m^2c^4$$

$$E=\pm\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$$

Pauli matrices: $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma}=\begin{pmatrix}k_z & k_x-ik_y\\ k_x+ik_y & -k_z\end{pmatrix}$$

$$\chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})\qquad \chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

Positive energy solutions: $E=\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\ 0\end{pmatrix}\qquad \chi^L_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\ \kappa_+\end{pmatrix}\qquad \kappa_z\equiv\frac{\hbar k_z c}{E+mc^2}$$

$$\chi^U_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\ 1\end{pmatrix}\qquad \chi^L_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\ -\kappa_z\end{pmatrix}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E+mc^2}$$

Pauli matrices: $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma}=\begin{pmatrix}k_z & k_x-ik_y\\ k_x+ik_y & -k_z\end{pmatrix}$$

$$\chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})\qquad \chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

Negative energy solutions: $E=-\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\ \kappa_+\end{pmatrix}\qquad \chi^L_\uparrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\ 0\end{pmatrix}\qquad \kappa_z\equiv\frac{\hbar k_z c}{E-mc^2}$$

$$\chi^U_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\ -\kappa_z\end{pmatrix}\qquad \chi^L_\downarrow(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\ 1\end{pmatrix}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E-mc^2}$$

What does this all mean?

Positive energy solutions: $E=\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$

$$\chi_\uparrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}\qquad \chi_\uparrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\\kappa_+\end{pmatrix}\qquad \kappa_z\equiv\frac{\hbar k_z c}{E+mc^2}$$

$$\chi_\downarrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}\qquad \chi_\downarrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\-\kappa_z\end{pmatrix}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E+mc^2}$$

For $\hbar c|\mathbf{k}|\ll mc^2$ $\quad E\approx mc^2+\frac{\hbar^2|\mathbf{k}|^2}{2m}$

$$\chi_\uparrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}\qquad \chi_\uparrow^L(\mathbf{k})\approx\mathcal{N}\begin{pmatrix}0\\0\end{pmatrix}$$

$$\chi_\downarrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}\qquad \chi_\downarrow^L(\mathbf{k})\approx\mathcal{N}\begin{pmatrix}0\\0\end{pmatrix}$$

Dirac equation for free particles

$$\chi_\uparrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}\qquad \chi_\uparrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\\kappa_+\end{pmatrix}$$

mc²

$$E=\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}\qquad \chi_\downarrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}\qquad \chi_\downarrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\-\kappa_z\end{pmatrix}$$

$$\kappa_z\equiv\frac{\hbar k_z c}{E+mc^2}\qquad \kappa_\pm\equiv\frac{\hbar k_\pm c}{E+mc^2}$$

-mc²

$$E=-\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}\qquad \chi_\uparrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_z\\\kappa_+\end{pmatrix}\qquad \chi_\uparrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\chi_\downarrow^U(\mathbf{k})=\mathcal{N}\begin{pmatrix}\kappa_-\\-\kappa_z\end{pmatrix}\qquad \chi_\downarrow^L(\mathbf{k})=\mathcal{N}\begin{pmatrix}0\\1\end{pmatrix}$$

Dirac equation for a free particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\\varphi^L\end{pmatrix}=\begin{pmatrix}mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c\\\mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\\varphi^L\end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

Other convenient notations in terms of 4×4 matrices

$$\boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix}\qquad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}\qquad H=\mathbf{p}\cdot\boldsymbol{\alpha}c+mc^2\beta$$

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\qquad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad I\equiv I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Dirac equation for Fermi particle in a scalar potential field

For free particle: $H = \mathbf{p}\cdot\boldsymbol{\alpha}c + mc^2\beta$

where: $\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$ $\quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and where:

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad I \equiv I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Dirac's suggestion for representing a scalar potential field:

$H = \mathbf{p}\cdot\boldsymbol{\alpha}c + mc^2\beta + V(\mathbf{r})I_4$ where $I_4 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$

Dirac equation for electron in the field of a H-like ion

$$H = \mathbf{p}\cdot\boldsymbol{\alpha}c + mc^2\beta + V(\mathbf{r})I_4$$

For H-like ion with nuclear charge Z:

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

Stationary state solutions:

$$\Psi(\mathbf{r},t) = \begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} e^{-iEt/\hbar}$$

$$\begin{pmatrix} mc^2 + V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 + V(r) \end{pmatrix} \begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix}$$

Dirac equation for electron in the field of a H-like ion -- continued

$$\underbrace{\begin{pmatrix} mc^2 + V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 + V(r) \end{pmatrix}}_{H} \begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix}$$

In order to find an efficient functional form for the eigenfunctions, it is helpful to determine the constants of the motion:

Total angular momentum: $\mathbf{J} = \mathbf{L} + \frac{\hbar}{2}\boldsymbol{\sigma}$ in terms of J_z and $\mathbf{J}^2$

Special K operator: $K = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L} + \hbar I_2 & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L} - \hbar I_2 \end{pmatrix}$

Dirac equation for electron in the field of a H-like ion -- continued

$$H = \begin{pmatrix} mc^2 + V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 + V(r) \end{pmatrix}$$

$$J_z = \begin{pmatrix} L_z + \frac{1}{2}\hbar\sigma_z & 0 \\ 0 & L_z + \frac{1}{2}\hbar\sigma_z \end{pmatrix}$$

$$\mathbf{J}^2 = \begin{pmatrix} \mathbf{L}^2 + \frac{3\hbar^2}{4}I_2 + \hbar\boldsymbol{\sigma}\cdot\mathbf{L} & 0 \\ 0 & \mathbf{L}^2 + \frac{3\hbar^2}{4}I_2 + \hbar\boldsymbol{\sigma}\cdot\mathbf{L} \end{pmatrix}$$

$$K = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L} + \hbar I_2 & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L} - \hbar I_2 \end{pmatrix}$$

Note that:

$$K^2 = \mathbf{J}^2 + \tfrac{1}{4}\hbar^2$$

Commutation relations:

$[H, \mathbf{J}^2] = 0 \quad [K, \mathbf{J}^2] = 0 \quad [K, J_z] = 0$

$[H, K] = 0,$ etc.

Dirac equation for electron in the field of a H-like ion -- continued

$$\begin{pmatrix} mc^2 + V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 + V(r) \end{pmatrix}\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = E\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix}$$

We can show that:

$$\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} g_{\kappa J}(r)\chi_{\kappa JM}(\hat{\mathbf{r}}) \\ if_{\kappa J}(r)\chi_{-\kappa JM}(\hat{\mathbf{r}}) \end{pmatrix}$$

More details about spin-angular functions:

Eigenfunctions of $\mathbf{J}^2$ and J_z : $|JM\rangle$

$\mathbf{J}^2|JM\rangle = \hbar^2 J(J+1)|JM\rangle$

$J_z|JM\rangle = \hbar M|JM\rangle$

$K^2|JM\rangle = \hbar^2\left(J(J+1) + \frac{1}{4}\right)|JM\rangle = \hbar^2\left(J + \frac{1}{2}\right)^2|JM\rangle$

Dirac equation for electron in the field of a H-like ion -- continued

Eigenvalues of K operator:

$$K = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L} + \hbar I_2 & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L} - \hbar I_2 \end{pmatrix}$$

$$K\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = -\hbar\kappa\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$K^2\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \hbar^2\left(J + \tfrac{1}{2}\right)^2\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Recall earlier discussion of "addition" of angular momentum

Addition of angular momentum - orbital angular momentum and spin angular momentum

Clebsch-Gordon coefficients

$$|JM, j_1 j_2\rangle = \sum_{m_1, m_2} |j_1 m_1, j_2 m_2\rangle \langle j_1 m_1, j_2 m_2 | JM, j_1 j_2\rangle$$

$$j_1 m_1 \Rightarrow l m_l \qquad J = l + \tfrac{1}{2}$$

$$j_2 m_2 \Rightarrow \tfrac{1}{2} m_s \qquad \text{or} \quad J = l - \tfrac{1}{2}$$

Clebsch-Gordon coefficients for this case:

$j_1 = l \qquad j_2 = s = \frac{1}{2}$

$$\left|\left(l+\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}\left|l\left(M-\tfrac{1}{2}\right); \tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}\left|l\left(M+\tfrac{1}{2}\right); \tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\left(l-\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = -\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}\left|l\left(M-\tfrac{1}{2}\right); \tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}\left|l\left(M+\tfrac{1}{2}\right); \tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

Another representation of the spin-angular function in terms of spherical harmonic functions and eigenvectors of σ_z

$$\left|\left(l+\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}Y_{l(M-\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}Y_{l(M+\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}0\\1\end{pmatrix}$$

$$\left|\left(l-\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = -\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}Y_{l(M-\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}Y_{l(M+\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}0\\1\end{pmatrix}$$

Consider: $\left(\boldsymbol{\sigma}\cdot\mathbf{L} + \hbar I_2\right)\left|JM; l\tfrac{1}{2}\right\rangle = \hbar\left(J(J+1) - l(l+1) + \tfrac{1}{4}\right)I_2\left|JM; l\tfrac{1}{2}\right\rangle$

$$= -\kappa\left|JM; l\tfrac{1}{2}\right\rangle$$

Possibilities:

$J = l + \frac{1}{2} \qquad \kappa = -l - 1$

$J = l - \frac{1}{2} \qquad \kappa = l$

Combinations:

κ	j	l
$\kappa = -1$	$j = \frac{1}{2}$	$l = 0$
$+1$	$\frac{1}{2}$	1
-2	$\frac{3}{2}$	1
$+2$	$\frac{3}{2}$	2
-3	$\frac{5}{2}$	2
$+3$	$\frac{5}{2}$	3

$$\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} g_{\kappa J}(r)\chi_{\kappa JM}(\hat{\mathbf{r}}) \\ if_{\kappa J}(r)\chi_{-\kappa JM}(\hat{\mathbf{r}}) \end{pmatrix}$$

Next time, we will show that the coupled radial equations take the form:

$$\frac{dg}{dr} + \frac{\kappa+1}{r} g = \frac{1}{\hbar c}\left(E + mc^2 - V(r)\right) f$$

$$\frac{df}{dr} - \frac{\kappa-1}{r} f = -\frac{1}{\hbar c}\left(E - mc^2 - V(r)\right) g$$

Question:

- How do these results reduce to the Schroedinger equation in the non-relativistic limit
- What new physics is contained more generally?

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