PHY 712 Quantum Mechanics
12-12:50 PM MWF Olin 103

Plan for Lecture 6:
Continue reading Chapter #5 in Shankar;
Quantum mechanical systems in 1-dim

1. **Particle bound in an infinite square well**
2. **Particle bound in a finite square well**
3. **Particle in the presence of a potential step**

PHY 741 Quantum Mechanics

MWF 12 PM - 12:50 PM	OPL 103	http://www.wfu.edu/~natalie/f17phy741/

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Course schedule

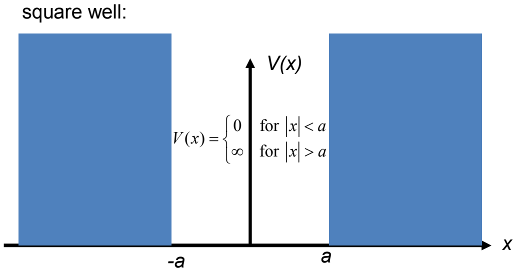
(Preliminary schedule -- subject to frequent adjustment.)

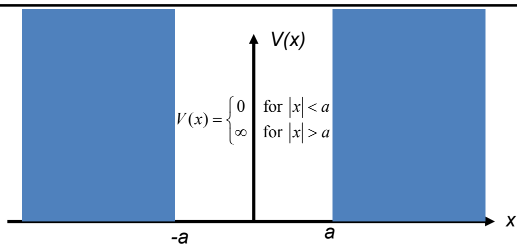
	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Review of basic principles	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Linear vector spaces	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Linear vector spaces	#3	9/6/2017
4	Mon, 9/04/2017	Chap. 4	Principles of Quantum Mechanics	#4	9/8/2017
5	Wed, 9/06/2017	Chap. 5	Examples in 1 dimension		
6	Fri, 9/08/2017	Chap. 6	Schrödinger equation in one-dimension	#5	9/13/2017
7	Mon, 9/11/2017				
8	Wed, 9/13/2017				

Energy eigenstates of the Schrödingder equation

$$H|E\rangle = E|E\rangle$$

Example: Particle of mass *m* confined within an infinite square well:





$$V(x)=\begin{cases}0 & \text{for } |x|<a\\ \infty & \text{for } |x|>a\end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\psi(x) \quad \text{for } |x|<a$$

$$\psi(x)=0 \quad \text{for } |x|>a$$

In this case, $E \geq 0$

$$\Rightarrow \psi(x)=C_1 e^{i(\sqrt{2mE}/\hbar)x}+C_2 e^{-i(\sqrt{2mE}/\hbar)x}$$

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$$\Rightarrow \psi(x)=C_1 e^{i(\sqrt{2mE}/\hbar)x}+C_2 e^{-i(\sqrt{2mE}/\hbar)x}$$

$$\text{or} \quad \psi(x)=C_3\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right)$$

$$\text{or} \quad \psi(x)=C_4\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$$

Satisfying the boundary conditions:

$$\psi(\pm a)=0 \qquad \Rightarrow \sin\left(\frac{\sqrt{2mE}}{\hbar}a\right)=0$$

$$\text{or} \qquad \cos\left(\frac{\sqrt{2mE}}{\hbar}a\right)=0$$

Satisfying the boundary conditions:

$$\psi(\pm a)=0 \qquad \Rightarrow \sin\left(\frac{\sqrt{2mE}}{\hbar}a\right)=0 \rightarrow \frac{\sqrt{2mE}\,a}{\hbar}=n\pi \quad n=1,2,3...$$

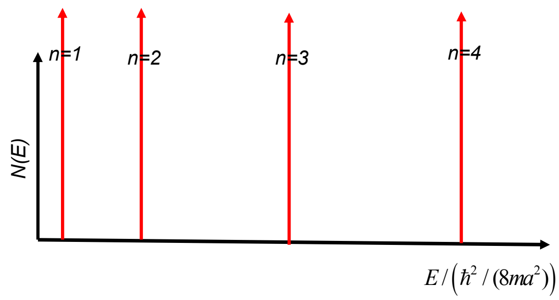
$$\text{or} \qquad \cos\left(\frac{\sqrt{2mE}}{\hbar}a\right)=0 \rightarrow \frac{\sqrt{2mE}\,a}{\hbar}=\frac{(2n+1)\pi}{2} \quad n=1,2,3...$$

Summary of results:

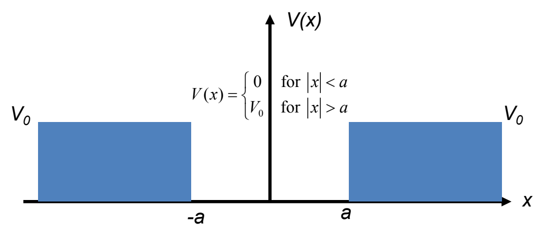
$$E=\frac{\hbar^2\pi^2 n^2}{8ma^2} \qquad \text{For } n \text{ odd:} \quad \psi(x)=\sqrt{\frac{1}{a}}\cos\left(\frac{n\pi x}{2a}\right)$$

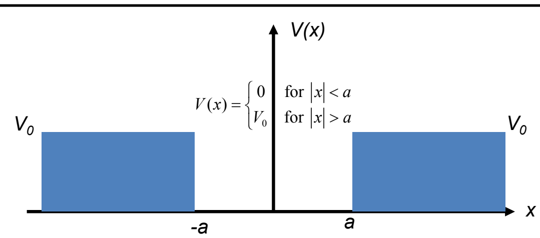
$$\text{For } n \text{ even:} \quad \psi(x)=\sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right)$$

Summary: Energy eigenvalue spectrum for a particle of mass m confined within an infinite square well of width $2a$:



Example: Particle of mass m confined within an finite square well:





$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \quad \text{for } |x| < a$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - V_0)\psi(x) \quad \text{for } |x| > a$$

Note that either $\psi(-x) = \psi(x)$ (even solution)

or $\psi(-x) = -\psi(x)$ (odd solution)

First consider the case $E>V_0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\psi(x) \quad \text{for } |x|<a$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=\left(E-V_0\right)\psi(x) \quad \text{for } |x|>a$$

Even solution:

For $|x|\le a$: $\quad \psi(x)=C_1\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$

For $x\ge a$: $\quad \psi(x)=C_2\cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}x\right)+C_3\sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}x\right)$

For $x\le -a$: $\quad \psi(x)=C_2\cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}x\right)-C_3\sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}x\right)$

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For $|x|\le a$: $\quad \psi(x)=C_1\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$

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Satisfying boundary values at $x=a$:

$$C_1\cos\left(\frac{\sqrt{2mE}}{\hbar}a\right)-C_2\cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right)-C_3\sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right)=0$$

$$\sqrt{\frac{E}{E-V_0}}C_1\sin\left(\frac{\sqrt{2mE}}{\hbar}a\right)-C_2\sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right)+C_3\cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right)=0$$

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Satisfying boundary values at $x=a$:

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Equations for coefficients:

$$\begin{pmatrix} \cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right) & \sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right) \\ \sin\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right) & -\cos\left(\frac{\sqrt{2m\left(E-V_0\right)}}{\hbar}a\right) \end{pmatrix}\begin{pmatrix} C_2/C_1 \\ C_3/C_1 \end{pmatrix}=\begin{pmatrix} \cos\left(\frac{\sqrt{2mE}}{\hbar}a\right) \\ \sqrt{\frac{E}{E-V_0}}\sin\left(\frac{\sqrt{2mE}}{\hbar}a\right) \end{pmatrix}$$

➔solution for all $E>V_0$ → continuum spectrum
similar solution for odd solutions

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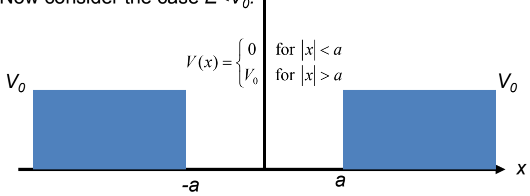
Now consider the case $E<V_0$:



$$V(x) = \begin{cases} 0 & \text{for } |x| < a \\ V_0 & \text{for } |x| > a \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \quad \text{for } |x| < a$$

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Even solutions:

For $|x| \le a$: $\psi(x) = C_1 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$

For $|x| \ge a$: $\psi(x) = C_2 e^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}|x|}$

Boundary conditions:

$$C_1 \cos\left(\frac{\sqrt{2mE}}{\hbar}a\right) = C_2 e^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}a}$$

$$C_1 \sin\left(\frac{\sqrt{2mE}}{\hbar}a\right) = \sqrt{\frac{V_0-E}{E}}\, C_2 e^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}a}$$

Finite potential well – even parity bound solutions

Boundary conditions:

$$C_1 \cos\left(\frac{\sqrt{2mE}}{\hbar}a\right) = C_2 e^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}a}$$

$$C_1 \sin\left(\frac{\sqrt{2mE}}{\hbar}a\right) = \sqrt{\frac{V_0-E}{E}}\, C_2 e^{-\frac{\sqrt{2m(V_0-E)}}{\hbar}a}$$

$$\tan\left(\frac{\sqrt{2mE}}{\hbar}a\right) = \sqrt{\frac{V_0-E}{E}}$$

Similarly, we can analyze the odd-parity bound states:

$$\cot\left(\frac{\sqrt{2mE}}{\hbar}a\right) = -\sqrt{\frac{V_0-E}{E}}$$

Solution of transcendental equations:

$$\tan\left(\frac{\sqrt{2mE}}{\hbar}a\right)=\sqrt{\frac{V_0-E}{E}} \quad \Rightarrow \tan\left(\frac{\sqrt{2mV_0}a}{\hbar}\sqrt{\frac{E}{V_0}}\right)=\sqrt{\frac{1-E/V_0}{E/V_0}}$$

$$\cot\left(\frac{\sqrt{2mE}}{\hbar}a\right)=-\sqrt{\frac{V_0-E}{E}} \quad \Rightarrow \cot\left(\frac{\sqrt{2mV_0}a}{\hbar}\sqrt{\frac{E}{V_0}}\right)=-\sqrt{\frac{1-E/V_0}{E/V_0}}$$

Let $u\equiv\frac{\sqrt{2mV_0}a}{\hbar}$ $\quad \epsilon\equiv\sqrt{\frac{E}{V_0}}$

$$\tan(u\epsilon)=\sqrt{\frac{1-\epsilon^2}{\epsilon^2}}$$

or $\quad \cot(u\epsilon)=-\sqrt{\frac{1-\epsilon^2}{\epsilon^2}}$

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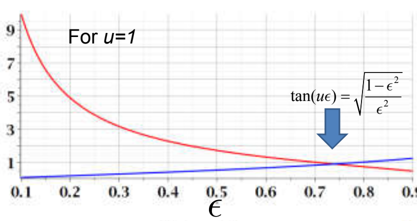
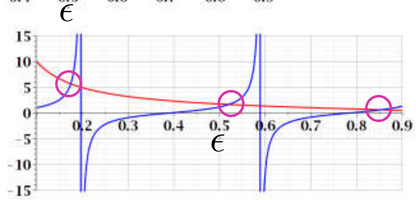


For *u=8*



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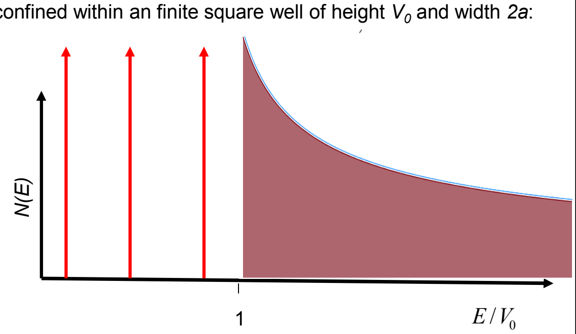
Summary: Energy eigenvalue spectrum for a particle of mass *m* confined within an finite square well of height V_0 and width *2a*:



Discrete eigenvalues Continuum eigenvalues

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