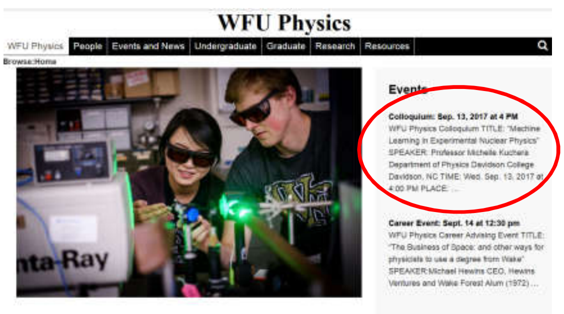
PHY 712 Quantum Mechanics
12-12:50 PM MWF Olin 103

Plan for Lecture 8:
Start reading Chapter #7 in Shankar;
Eigenstates of the harmonic oscillator

1. **Solution of the differential equation**
2. **Operator formalism**



WFU Physics Colloquium

TITLE: "Machine Learning in Experimental Nuclear Physics"

SPEAKER: Professor Michelle Kuchera
Department of Physics
Davidson College
Davidson, NC

TIME: Wed. Sep. 13, 2017 at 4:00 PM

PLACE: George P. Williams, Jr. Lecture Hall, (Olin 101)

ABSTRACT

The atomic nucleus was discovered by Ernest Rutherford in 1911. Over 100 years later, as the limits of nuclear existence are beginning to be explored, we are still working toward a complete understanding of the structure and properties of all nuclei. Advanced accelerators and detectors are used to study reactions involving rare nuclei. This talk will discuss the computational challenges and methods of "big data" detector systems.

PHY 741 Quantum Mechanics

MWF 12 PM - 12:50 PM | OPL 103 | http://www.wfu.edu/~natalie/f17phy741/

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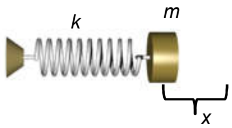
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Review of basic principles	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Linear vector spaces	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Linear vector spaces	#3	9/6/2017
4	Mon, 9/04/2017	Chap. 4	Principles of Quantum Mechanics	#4	9/8/2017
5	Wed, 9/06/2017	Chap. 5	Examples in 1 dimension		
6	Fri, 9/08/2017	Chap. 5	Schrödinger equation in one-dimension	#5	9/13/2017
7	Mon, 9/11/2017	Chap. 5	Schrödinger equation in one-dimension		
8	Wed, 9/13/2017	Chap. 7	Schrödinger equation in one-dimension	#6	9/15/2017
9	Fri, 9/15/2017				
10	Mon, 9/18/2017				

Quantum mechanics of a harmonic oscillator
from https://en.wikipedia.org/wiki/Simple_harmonic_motion

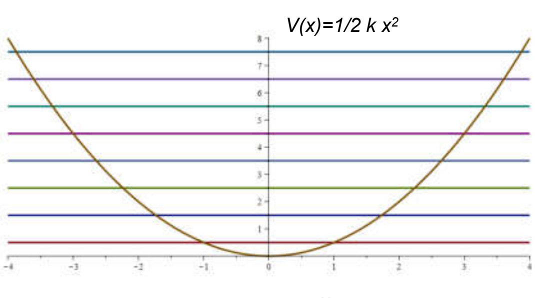


Classical trajectory:

$$x(t) = X_0 \cos(\omega t) \qquad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Oscillator potential:

$$V(x) = \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$



Comes from analysis of systems near equilibrium:

$$V(x) = V(x_0) + \left.\frac{dV}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x-x_0)^2 + \cdots$$

Eigenstates of the the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\psi = E\psi$$

Solutions to the differential equation -- let $x = by$

$$\frac{d^2\psi}{dy^2}+\frac{2mEb^2}{\hbar^2}\psi-\frac{m^2\omega^2b^4}{\hbar^2}y^2\psi=0$$

$$b=\left(\frac{\hbar}{m\omega}\right)^{1/2} \qquad \varepsilon=\frac{mEb^2}{\hbar^2}=\frac{E}{\hbar\omega}$$

In these units, the equation becomes:

$$\psi''+(2\varepsilon-y^2)\psi=0$$

Further transformation:

$$\psi(y)=u(y)\,e^{-y^2/2}$$

Equation for *u(y)*: $u''-2yu'+(2\varepsilon-1)u=0$

Hermite polynomial solutions for $u(y)=H_n(y)$

$$\frac{d^2H_n}{dy^2}-2y\frac{dH_n}{dy}+2nH_n=0$$

$$\Rightarrow (2\varepsilon-1-2n)H_n(y)=0 \qquad \Rightarrow \varepsilon_n=\frac{1}{2}+n$$

Complete solution including normalization

$$\psi_n(x)=\left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}x\right]$$

$$E_n=\hbar\omega\left(\frac{1}{2}+n\right) \qquad n=0,1,2,.....$$

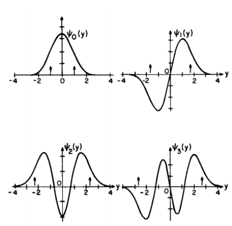


Figure 7.1. Normalized eigenfunctions for $n = 0, 1, 2,$ and 3. The small arrows at $|y|=(2n+1)^{1/2}$ stand for the classical turning points. Recall that $y=(m\omega/\hbar)^{1/2}x$.

Some useful relationships of Hermite polynomials:

$$H_n'(y) = 2nH_{n-1}$$

$$H_{n+1}(y) = 2yH_n - 2nH_{n-1}$$

$$\int_{-\infty}^{\infty} H_n(y)H_{n'}(y)\, e^{-y^2}\, dy = \delta_{nn'}(\pi^{1/2}2^n n!)$$

Useful matrix elements:

$$\langle n'|X|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} [\delta_{n',n+1}(n+1)^{1/2} + \delta_{n',n-1}n^{1/2}]$$

,

$$\langle n'|P|n\rangle = \left(\frac{m\omega\hbar}{2}\right)^{1/2} i[\delta_{n',n+1}(n+1)^{1/2} - \delta_{n',n-1}n^{1/2}]$$

Representation of the position and momentum operators in terms of the energy eigenstates of the harmonic oscillator: *n= 0 1 2 3 …..*

$$X \leftrightarrow \left(\frac{\hbar}{2m\omega}\right)^{1/2} \begin{bmatrix} 0 & 1^{1/2} & 0 & 0 & \cdots \\ 1^{1/2} & 0 & 2^{1/2} & 0 & \\ 0 & 2^{1/2} & 0 & 3^{1/2} & \\ 0 & 0 & 3^{1/2} & 0 & \\ \vdots & & & & \end{bmatrix}$$

$$P \leftrightarrow i\left(\frac{m\omega\hbar}{2}\right)^{1/2} \begin{bmatrix} 0 & -1^{1/2} & 0 & 0 & \cdots \\ 1^{1/2} & 0 & -2^{1/2} & 0 & \\ 0 & 2^{1/2} & 0 & -3^{1/2} & \\ 0 & 0 & 3^{1/2} & 0 & \\ \vdots & & & & \end{bmatrix}$$

Note that:

$[X, P] = i\hbar I = \imath n$

Average values and uncertainties

$$\langle n|X|n\rangle = 0 \qquad \langle n|X^2|n\rangle = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle n|P|n\rangle = 0 \qquad \langle n|P^2|n\rangle = \frac{\hbar m\omega}{2}(2n+1)$$

$$\langle n|\Delta X|n\rangle\langle n|\Delta p|n\rangle = \frac{\hbar}{2}(2n+1)$$

Analysis of the Harmonic Oscillator Hamiltonian in terms of raising and lowering operators

Define:

$$a=\left(\frac{m\omega}{2\hbar}\right)^{1/2}X+i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}P$$

$$a^{\dagger}=\left(\frac{m\omega}{2\hbar}\right)^{1/2}X-i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}P$$

It follows that: $[a, a^{\dagger}]=1$

Note that:

$$a^{\dagger}a=\frac{m\omega}{2\hbar}X^2+\frac{1}{2m\omega\hbar}P^2+\frac{i}{2\hbar}[X, P]$$

$$=\frac{H}{\hbar\omega}-\frac{1}{2}$$

so that

$$H=(a^{\dagger}a+1/2)\hbar\omega$$

Unitless operators:

$$\hat{H}=\frac{H}{\hbar\omega}=(a^{\dagger}a+1/2)$$

Eigenstates of $\hat{H}=\frac{H}{\hbar\omega}=(a^{\dagger}a+1/2)$

$$\hat{H}|\varepsilon\rangle=\varepsilon|\varepsilon\rangle$$

Note that: $[a, \hat{H}]=[a, a^{\dagger}a+1/2]=[a, a^{\dagger}a]=a$

$[a^{\dagger}, \hat{H}]=-a^{\dagger}$

$$\hat{H}a|\varepsilon\rangle=(a\hat{H}-[a, \hat{H}])|\varepsilon\rangle$$
$$=(a\hat{H}-a)|\varepsilon\rangle$$
$$=(\varepsilon-1)a|\varepsilon\rangle$$

$$\hat{H}a^{\dagger}|\varepsilon\rangle=(a^{\dagger}\hat{H}-[a^{\dagger}, H])|\varepsilon\rangle$$
$$=(a^{\dagger}\hat{H}+a^{\dagger})|\varepsilon\rangle$$
$$=(\varepsilon+1)a^{\dagger}|\varepsilon\rangle$$

$$\hat{H}a|\varepsilon\rangle = (a\hat{H} - [a,\hat{H}])|\varepsilon\rangle$$
$$= (a\hat{H} - a)|\varepsilon\rangle$$
$$= (\varepsilon - 1)a|\varepsilon\rangle$$
$$\Downarrow$$
$$a|\varepsilon\rangle = C_\varepsilon|\varepsilon - 1\rangle$$

$$\hat{H}a^\dagger|\varepsilon\rangle = (a^\dagger\hat{H} - [a^\dagger, H])|\varepsilon\rangle$$
$$= (a^\dagger\hat{H} + a^\dagger)|\varepsilon\rangle$$
$$= (\varepsilon + 1)a^\dagger|\varepsilon\rangle$$
$$\Downarrow$$
$$a^\dagger|\varepsilon\rangle = D_\varepsilon|\varepsilon + 1\rangle$$

Argue that the eigenvalues have a sequence and can be labeled with energy *n=0,1,2,3*

$$a|\varepsilon_0\rangle = 0$$
$$a^\dagger a|\varepsilon_0\rangle = 0$$
$$(\hat{H} - 1/2)|\varepsilon_0\rangle = 0 \;\Rightarrow\; \hat{H}|\varepsilon_0\rangle = \tfrac{1}{2}|\varepsilon_0\rangle \;\Rightarrow\; \varepsilon_0 = \tfrac{1}{2}$$

This implies

$$|\varepsilon_n\rangle \equiv |n\rangle \qquad \text{Where } \hat{H}|n\rangle = \left(\frac{1}{2} + n\right)|n\rangle$$

Determining coefficients:

$$a|\varepsilon\rangle = C_\varepsilon|\varepsilon - 1\rangle \qquad a^\dagger|\varepsilon\rangle = D_\varepsilon|\varepsilon + 1\rangle$$

$$\left(\hat{H} - \frac{1}{2}\right)|n\rangle = a^\dagger a|n\rangle = n|n\rangle$$

$$\langle n|a^\dagger a|n\rangle = \langle an|an\rangle = |C_n|^2\langle n-1|n-1\rangle = n\langle n|n\rangle$$

$$\Rightarrow C_n = \sqrt{n} \qquad \text{similarly, } D_n = \sqrt{n+1}$$

Summary of results

$$H|n\rangle = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)|n\rangle = \hbar\omega\left(\frac{1}{2} + n\right)|n\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$