PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103
Plan for Lecture 10:

Continue reading Chapter 3 & 6

1. **Constants of the motion**
2. **Conserved quantities**
3. **Legendre transformations**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

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	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism		

Public Talk – ΦBK Lecturer -- Mon. 9/24/2018 at 7 PM

Exploring Space for Earth: Earth's Vital Signs Revealed with Dava Newman

Professsor Dava Newman is the former Deputy Administrator of NASA, an MIT Apollo Professor of Astronautics at the Massachusetts Institute of Technology and an expert in space technology and policy. Her public lecture offers an orbital view of planet Earth's interconnected systems through supercomputer data visualizations and stories to demonstrate risks, actions and solutions.

Monday, September 24 at 7:00pm

Porter Byrum Welcome Center

"Human Exploration from Earth to Mars: Becoming Interplanetary"

Physics Colloquium:
Tue. 9/25/2018 at 4 PM
Olin 101

Prof. Dava Newman
ΦBK Lecturer from MIT

Recent space science missions to Pluto and Jupiter, the discovery of thousands of exoplanets, and orbital missions to monitor Spaceship Earth will be highlighted. Humanity will become interplanetary, and is on a journey to Mars. We are closer to reaching the Red Planet with human explorers than we have ever been in our history. Space agencies, academia and industry are working right now on the technologies and missions that will enable human "boots on Mars" in the 2030s. We are testing advanced technologies for the next giant leaps of exploration. From solar electric propulsion to cutting edge life support systems, advanced space suits, to the first crops grown in space, the journey to Mars is already unfolding in tangible ways today for tomorrow.

A three-stage plan will be highlighted – from missions close to Earth involving commercial partners and the International Space Station, advancing to missions in Earth–Moon orbit, or deep space, and finally moving on to Mars, where explorers will be practically independent from spaceship Earth. The innovation required to realize humanity becoming interplanetary cuts across science, human exploration and technology.

Fundamentally, education, knowledge and access are the keys to exploring our solar system, Spaceship Earth, and ourselves. The urgency of education about our own planet is shown through supercomputer data visualizations accessible through online open platforms. The presentation concludes with an inclusive message on STEAMD (science-technology-engineering-arts-math-design) about changing the conversation to include everyone: the artists, designers, poets and makers. We are all astronauts on Spaceship Earth!

Eleanor Roosevelt once said that the "future belongs to those who believe in the beauty of their dreams."

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

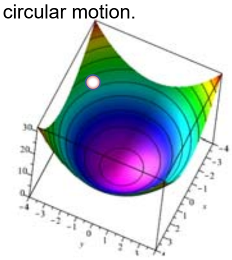
$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = \text{(constant)}$$

Consider a particle of mass *m* moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+4c^2\left(x\dot{x}+y\dot{y}\right)^2\right) - mgc\left(x^2+y^2\right)$$

Transform to polar coordinates;

$x = r\cos\phi \qquad y = r\sin\phi$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2+4c^2r^2\dot{r}^2\right) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad 0 - \frac{d}{dt}mr^2\dot{\phi} = 0$$

$$\Rightarrow \quad \text{Let } mr^2\dot{\phi} \equiv \ell_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2+4c^2r^2\dot{r}^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

$$\frac{\ell_z^{\ 2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Now consider the case where initially the particle is moving in a circle at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Consider small perturbation to the motion: $r = r_0 + \delta r$

$$\frac{\ell_z^{\ 2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Consider small perturbation to the motion: $r = r_0 + \delta r$
where initially the particle is moving in a circle
at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1+2c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1+2c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1+2c^2r_0^2}}t + \alpha\right)$$

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \qquad \Rightarrow m\dot{x} \equiv p_x \ \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \qquad \Rightarrow m\dot{y} \equiv p_y \ \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

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Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt} mr^2\dot{\varphi} = 0 \qquad \Rightarrow mr^2\dot{\varphi} \equiv p_\varphi \ \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{r} = mr\dot{\varphi}^2 - \frac{\partial V}{\partial r} = \frac{{p_\varphi}^2}{mr^3} - \frac{\partial V}{\partial r}$$

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Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that:

$$\frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

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Additional constant of the motion:

If $\frac{\partial L}{\partial t}=0;$

then: $\frac{d}{dt}\left(L-\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right)=\frac{\partial L}{\partial t}=0$

$\Rightarrow L-\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma=-E$ (constant)

Example 1: one-dimensional potential:

$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-V(z)$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-V(z)-m\dot{x}^2-m\dot{y}^2-m\dot{z}^2\right)=0$

$\Rightarrow -\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+V(z)\right)=-E$ (constant)

For this case, we also have $m\dot{x}\equiv p_x$ and $m\dot{y}\equiv p_y$

$\Rightarrow E=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{1}{2}m\dot{z}^2+V(z)$

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t}=0;$

then: $\frac{d}{dt}\left(L-\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right)=\frac{\partial L}{\partial t}=0$

$\Rightarrow L-\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma=-E$ (constant)

Example 2: Motion in a central potential

$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-V(r)$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-V(r)-m\dot{r}^2-mr^2\dot{\phi}^2\right)=0$

$\Rightarrow -\left(\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)+V(r)\right)=-E$ (constant)

For this case, we also have $mr^2\dot{\varphi}\equiv p_\varphi$

$\Rightarrow E=\frac{p_\varphi^2}{2mr^2}+\frac{1}{2}m\dot{r}^2+V(r)$

Other examples

$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{q}{2c}B_0\left(-\dot{x}y+\dot{y}x\right)$

$\frac{\partial L}{\partial z}=0 \qquad \Rightarrow m\dot{z}=p_z$ (constant)

$$E=\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma-L$$

$$=m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{q}{2c}B_0\left(-\dot{x}y+\dot{y}x\right)$$

$$-\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-\frac{q}{2c}B_0\left(-\dot{x}y+\dot{y}x\right)$$

$$=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\frac{p_z^2}{2m}$$

Other examples

$$L = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{\partial L}{\partial z}=0 \qquad \Rightarrow m\dot{z}=p_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial x}=0 \qquad \Rightarrow m\dot{x}=p_x \quad \text{(constant)}$$

$$E=\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$-\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{q}{c}B_0\dot{x}y$$

$$= \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \tfrac{1}{2}m\dot{y}^2 + \frac{p_x^2}{2m}+\frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x,y) \quad \Leftrightarrow \quad x(y,z)\,???$$

$$z(x,y) \;\Rightarrow\; dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \;\Rightarrow\; dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\text{But}: \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y}$$

Simple change of variables -- continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} \qquad \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{(\partial z/\partial x)_y}$$

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Simple change of variables -- continued:

Example:

$$z(x,y) = e^{x^2+y}$$

$$x(y,z) = (\ln z - y)^{1/2}$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\left(\frac{\partial x}{\partial y}\right)_z \stackrel{?}{=} -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} \qquad \left(\frac{\partial x}{\partial z}\right)_y \stackrel{?}{=} \frac{1}{(\partial z/\partial x)_y}$$

$$-\frac{1}{2(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \qquad \frac{1}{2z(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

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Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y}\right)_x$

Define new function

$$w(u,y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

For $w = z - ux,$ $\quad dw = dz - udx - xdu = udx + vdy - udx - xdu$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

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For thermodynamic functions:

$$\text{Internal energy}: \quad U = U(S,V)$$
$$dU = TdS - PdV$$
$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$
$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$
$$\text{Enthalpy}: \quad H = H(S,P) = U + PV$$
$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$
$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$
$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma}dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}d\dot{q}_\sigma\right) - \frac{\partial L}{\partial t}dt$$

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$