

PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF Olin 103

Plan for Lecture 11:

Continue reading Chapter 3 & 6

1. Constructing the Hamiltonian
2. Hamilton's canonical equation
3. Examples

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

>>>>

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018
12	Wed, 9/26/2018	Chap. 3 and 6	Canonical transformations		

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Department of Physics

SPS Meeting and Career Advising Event



Professor Dava Newman

Apollo Program Professor of Astronautics
Department of Aeronautics and Astronautics
Harvard-MIT Health, Sciences, and Technology
Phi Beta Kappa Visiting Scholar

12:15 PM today in Olin 105

9/24/2018

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Public Talk – ΦBK Lecturer -- Mon. 9/24/2018 at 7 PM



Exploring Space for Earth: Earth's Vital Signs Revealed with Dava Newman

Professor Dava Newman is the former Deputy Administrator of NASA, an MIT Apollo Professor of Astronautics at the Massachusetts Institute of Technology and an expert in space technology and policy. Her public lecture offers an orbital view of planet Earth's interconnected systems through supercomputer data visualizations and stories to demonstrate risks, actions and solutions.

Monday, September 24 at 7:00pm

Porter Byrum Welcome Center

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"Human Exploration from Earth to Mars: Becoming Interplanetary"

Physics Colloquium:
Tue. 9/25/2018 at 4 PM
Olin 101



Prof. Dava Newman
ΦBK Lecturer from MIT

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Recent space science missions to Pluto and Jupiter, the discovery of thousands of exoplanets, and orbital missions to monitor Spaceship Earth will be highlighted. Humanity will become interplanetary, and is on a journey to Mars. We are closer to reaching the Red Planet with human explorers than we have ever been in our history. Space agencies, academia and industry are working right now on the technologies and missions that will enable human "boots on Mars" in the 2030s. We are testing advanced technologies for the next giant leaps of exploration. From solar electric propulsion to cutting edge life support systems, advanced space suits, to the first crops grown in space, the journey to Mars is already unfolding in tangible ways today for tomorrow.

A three-stage plan will be highlighted -- from missions close to Earth involving commercial partners and the International Space Station, advancing to missions in Earth-Moon orbit, or deep space, and finally moving on to Mars, where explorers will be practically independent from spaceship Earth. The innovation required to realize humanity becoming interplanetary cuts across science, human exploration and technology.

Fundamentally, education, knowledge and access are the keys to exploring our solar system, Spaceship Earth, and ourselves. The urgency of education about our own planet is shown through supercomputer data visualizations accessible through online open platforms. The presentation concludes with an inclusive message on STEAMD (science-technology-engineering-arts-math)-designed about changing the conversation to include everyone: the artists, designers, poets and makers. We are all astronauts on Spaceship Earth!

Eleanor Roosevelt once said that the "future belongs to those who believe in the beauty of their dreams."

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

⇒ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

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Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

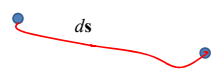
$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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Direct application of Hamiltonian's principle using the Hamiltonian function --

Generalized coordinates:
 $q_\sigma(\{x_i\})$ Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$\Rightarrow \text{Minimization integral: } S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

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Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \dot{\delta q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma} \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

Canonical equations

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma \dot{\delta q}_\sigma p_\sigma \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left(\sum_\sigma \delta q_\sigma \dot{p}_\sigma \right) dt$$

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Constants of the motion in Hamiltonian formalism

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

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Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

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Example 1: one - dimensional potential :

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$p_x = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

$$\text{Constants : } p_x, p_y, H$$

$$\text{Equations of motion : } \frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \frac{dp_z}{dt} = -\frac{dV}{dz}$$

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Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$p_r = m\dot{r} \quad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) \right) \\ = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants: p_ϕ, H

Equations of motion:

$$\frac{dr}{dt} = \frac{p_r}{m} \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

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Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})$$

$$H = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + Mgh \cos \beta$$

$$H = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh \cos \beta$$

Constants: p_α, p_γ, H

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Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + y\dot{x})$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y \right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x \right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: p_z, H

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Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : p_z, H

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y\right)$$

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Canonical equations of motion for constant magnetic field
-- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x\right) = \frac{qB_0}{2c} \frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y\right) = -\frac{qB_0}{2c} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

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General treatment of particle of mass m and charge q moving in 3 dimensions in a potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$:

Lagrangian: $L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

Hamiltonian: $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t)$

$$H(\mathbf{r}, \mathbf{p}, t) = \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t)$$

$$= \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t)$$

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Poisson brackets:

Recall:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

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Poisson brackets -- continued:

For an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F, G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right) = -[G, F]_{PB}$$

$$\text{So that : } \frac{dF}{dt} = [F, H]_{PB} + \frac{\partial F}{\partial t}$$

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Poisson brackets -- continued:

$$[F, G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right) = -[G, F]_{PB}$$

Examples :

$$[x, x]_{PB} = 0 \quad [x, p_x]_{PB} = 1 \quad [x, p_y]_{PB} = 0$$

$$[L_x, L_y]_{PB} = L_z$$

Liouville theorem

Let $D \equiv$ density of particles in phase space :

$$\frac{dD}{dt} = [D, H]_{PB} + \frac{\partial D}{\partial t} = 0$$

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Phase space

Phase space is defined as the set of all coordinates and momenta of a system :

$$(\{q_\sigma(t)\}, \{p_\sigma(t)\})$$

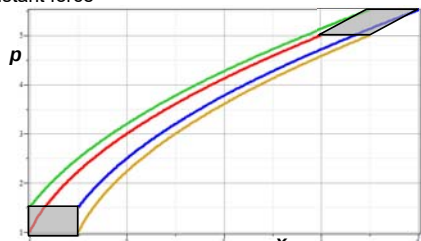
For a d dimensional system with N particles, the phase space corresponds to $2dN$ degrees of freedom.

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Phase space diagram for one-dimensional motion due to constant force



$$H(x, p) = \frac{p^2}{2m} - F_0 x \quad \dot{p} = F_0 \quad \dot{x} = \frac{p}{m}$$

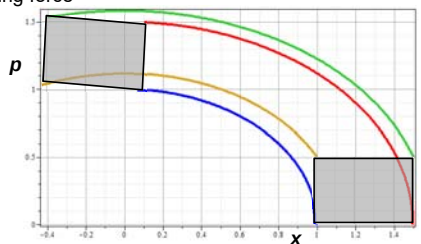
$$p_i(t) = p_{0i} + F_0 t \quad x_i(t) = x_{0i} + \frac{p_{0i}}{m} t + \frac{1}{2} F_0 t^2$$

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Phase space diagram for one-dimensional motion due to spring force



$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \quad \dot{p} = -m \omega^2 x \quad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i} \cos(\omega t + \theta_{0i}) \quad x_i(t) = \frac{p_{0i}}{m \omega} \sin(\omega t + \theta_{0i})$$

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Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dD}{dt} = \sum_{\sigma} \left(\frac{\partial D}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial D}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) + \frac{\partial D}{\partial t}$$

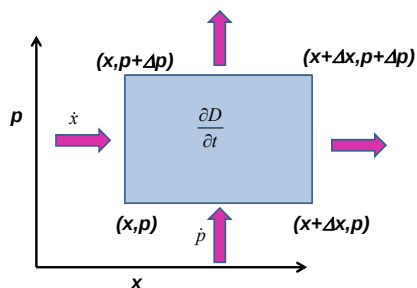
According to Liouville's theorem: $\frac{dD}{dt} = 0$

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Liouville's theorem

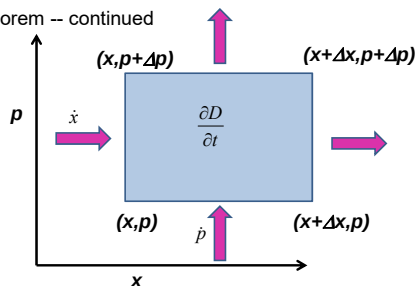


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Liouville's theorem -- continued



$$\begin{aligned} \frac{\partial D}{\partial t} &\Rightarrow \text{time rate of change of particles within volume} \\ &= \text{time rate of particle entering minus particles leaving} \\ &= -\frac{\partial D}{\partial x} \dot{x} - \frac{\partial D}{\partial p} \dot{p} \end{aligned}$$

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Liouville's theorem -- continued

$$\frac{\partial D}{\partial t} = -\frac{\partial D}{\partial x} \dot{x} - \frac{\partial D}{\partial p} \dot{p}$$

$$\frac{\partial D}{\partial t} + \frac{\partial D}{\partial x} \dot{x} + \frac{\partial D}{\partial p} \dot{p} = 0 = \frac{dD}{dt}$$

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Review:
Liouville's theorem:
Imagine a collection of particles obeying the Canonical equations of motion in phase space.

Let D denote the "distribution" of particles in phase space :

$$D = D(\{q_1 \cdots q_{3N}\}, \{p_1 \cdots p_{3N}\}, t)$$

Liouville's theorem shows that :

$$\frac{dD}{dt} = 0 \quad \Rightarrow D \text{ is constant in time}$$

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Proof of Liouville's theorem:

Continuity equation :

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v}D)$$

Note : in this case, the velocity is the $6N$ dimensional vector :

$$\mathbf{v} = (\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dots, \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \dots, \dot{\mathbf{p}}_N)$$

We also have a $6N$ dimensional gradient :

$$\nabla = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \dots, \nabla_{\mathbf{r}_N}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \dots, \nabla_{\mathbf{p}_N})$$

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$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v}D)$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial}{\partial q_j} (\dot{q}_j D) + \frac{\partial}{\partial p_j} (\dot{p}_j D) \right]$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial D}{\partial q_j} \dot{q}_j + \frac{\partial D}{\partial p_j} \dot{p}_j \right] - D \sum_{j=1}^{3N} \left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right]$$

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j} \right) = 0$$

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$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N} \left[\frac{\partial D}{\partial q_j} \dot{q}_j + \frac{\partial D}{\partial p_j} \dot{p}_j \right] - D \sum_{j=1}^{3N} \left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right] \quad 0$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N} \left[\frac{\partial D}{\partial q_j} \dot{q}_j + \frac{\partial D}{\partial p_j} \dot{p}_j \right]$$

$$\Rightarrow \frac{\partial D}{\partial t} + \sum_{j=1}^{3N} \left[\frac{\partial D}{\partial q_j} \dot{q}_j + \frac{\partial D}{\partial p_j} \dot{p}_j \right] = \frac{dD}{dt} = 0$$

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$$\frac{dD}{dt} = 0$$

Importance of Liouville's theorem to statistical mechanical analysis:

In statistical mechanics, we need to evaluate the probability of various configurations of particles. The fact that the density of particles in phase space is constant in time, implies that each point in phase space is equally probable and that the time average of the evolution of a system can be determined by an average of the system over phase space volume.

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