PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 13:

Finish reading Chapter 6

1. **Virial theorem**
2. **Canonical transformations**
3. **Hamilton-Jacobi formalism**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

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	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018
12	Wed, 9/26/2018	Chap. 3 and 6	Liouville theorem	#10	10/3/2018
13	Fri, 9/28/2018	Chap. 3 and 6	Canonical transformations		
14	Mon, 10/1/2018	Chap. 4	Small oscillations about equilibrium		
15	Wed, 10/3/28/2018	Chap. 4	Normal modes of vibration		
16	Fri, 10/6/2018	Chap. 1-4, 6	Review		

Virial theorem (Clausius ~ 1860)

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Proof:

Define: $A \equiv \sum_\sigma \mathbf{p}_\sigma \cdot \mathbf{r}_\sigma$

$$\frac{dA}{dt} = \sum_\sigma (\dot{\mathbf{p}}_\sigma \cdot \mathbf{r}_\sigma + \mathbf{p}_\sigma \cdot \dot{\mathbf{r}}_\sigma) = \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma + 2T$$

Because $\dot{\mathbf{p}}_\sigma = \mathbf{F}_\sigma$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dA(t)}{dt} dt = \frac{A(\tau) - A(0)}{\tau} \Rightarrow 0$$

Note that this implies that the motion is bounded

$$\Rightarrow \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle = 0$$

Examples of the Virial Theorem

$$2\langle T \rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Harmonic oscillator:

$$\mathbf{F} = -kx\hat{\mathbf{x}} \qquad T = \frac{1}{2}m\dot{x}^2 \qquad \langle m\dot{x}^2 \rangle = \langle kx^2 \rangle$$

Check: for $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \alpha\right)$

$$\langle m\dot{x}^2 \rangle = \frac{1}{2}kA^2 = \langle kx^2 \rangle$$

Examples of the Virial Theorem

$$2\langle T \rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Circular orbit due to gravitational field of massive object:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad T = \frac{1}{2}mr^2\omega^2 \qquad \langle mr^2\omega^2 \rangle = \left\langle \frac{GMm}{r} \right\rangle$$

Check: for $r\omega^2 = \frac{GM}{r^2}$ $\qquad \Rightarrow \langle mr^2\omega^2 \rangle = \left\langle \frac{GMm}{r} \right\rangle$

Hamiltonian formalism and the canonical equations of motion:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Notion of "Canonical" transformations

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$
$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

For some $\tilde{H}$ and F, using Legendre transformations

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f} \left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = 0$$

$$\delta \int_{t_i}^{t_f} \left[\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = \int_{t_i}^{t_f} \left[\frac{d}{dt}\delta F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt$$

$$= \delta F\left(t_f\right) - \delta F\left(t_i\right) = 0 \qquad \text{and} \quad \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

Some relations between old and new variables:

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

$$\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right) = \sum_\sigma\left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma\right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$

$$\Rightarrow \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) = H\left(\{q_\sigma\},\{p_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose}: \ \dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} = 0 \ \text{ and } \ \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose}: \quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma \left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution:

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears:

Assume $\{Q_\sigma\},\{P_\sigma\},\widetilde{H}$ are constants; choose $\widetilde{H} = 0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note: S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \cancelto{0}{P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t)} + \frac{d}{dt}\left(-\sum_\sigma \cancelto{0}{P_\sigma Q_\sigma} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \cancelto{0}{P_\sigma \dot{Q}_\sigma} - \tilde{H}(\cancelto{0}{\{Q_\sigma\},\{P_\sigma\},t}) + \frac{d}{dt}\left(-\sum_\sigma \cancelto{0}{P_\sigma Q_\sigma} + S(\{q_\sigma\},\{P_\sigma\},t) \right)$$

$$\int_{t_i}^{t_f} \left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) \right) dt = \int_{t_i}^{t_f} \left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right) \right) dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Example: $\quad H(\{q\},\{p\},t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$

Hamilton-Jacobi Eq: $\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $\quad S(q,t) \equiv W(q) - Et \qquad (E \text{ constant})$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $\quad S(q,t) \equiv W(q) - Et \qquad (E \text{ constant})$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$
$$= \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) + C$$
$$S(q,E,t) = \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - Et$$
$$\frac{\partial S}{\partial E} = Q = \frac{1}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - t$$
$$\Rightarrow q(t) = \frac{\sqrt{2mE}}{m\omega} \sin(\omega(t+Q))$$

Another example of Hamilton Jacobi equations

Example: $H(\{q\},\{p\},t) = \frac{p^2}{2m} + mgy$

Assume $y(0) = h; \quad p(0) = 0$

Hamilton-Jacobi Eq: $H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy + \frac{\partial S}{\partial t} = 0$$

Assume: $S(y,t) \equiv W(y) - Et$ $\quad$ (E constant)

Example: $H(\{q\},\{p\},t) = \frac{p^2}{2m} + mgy$

Assume $y(0) = h; \quad p(0) = 0$

$$\frac{1}{2m}\left(\frac{dW}{dy}\right)^2 + mgy = E \equiv mgh$$
$$W(y) = m\int_y^h \sqrt{2g(h-y')}\, dy' = \frac{2}{3} m\sqrt{2g}\,(h-y)^{3/2}$$
$$S(y,t) = W(y) - Et = \frac{2}{3} m\sqrt{2g}\,(h-y)^{3/2} - mght$$

Check action:

For this case: $y(t) = h - \frac{1}{2}gt^2$

$$S = \int_0^t \left(\frac{1}{2}m\dot{y}^2 - mgy\right)dt' = \frac{1}{3}mg^2t^3 - mght$$

$$S(y,t) = W(y) - Et = \frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2} - mght$$

Recap --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Hamiltonian picture

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$\Rightarrow$ Coupled first order differential equations for

$q_\sigma(t)$ and $p_\sigma(t)$