

**PHY 711 Classical Mechanics and
Mathematical Methods
10:10:50 AM MWF Olin 103**

Course schedule					
(Preliminary schedule -- subject to frequent adjustment.)					
>>>>>					
Date	F&W Reading	Topic	Assignment	Due	
1 Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018	
Wed, 8/29/2018	No class				
2 Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018	
3 Mon, 9/3/2018	Chap. 1	Scattering theory	#3	9/10/2018	
4 Wed, 9/5/2018	Chap. 1	Scattering theory	#4	9/12/2018	
5 Fri, 9/7/2018	Chap. 2	Non-inertial coordinate systems	#5	9/12/2018	
6 Mon, 9/10/2018	Chap. 3	Calculus of Variation	#6	9/12/2018	
7 Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018	
Fri, 9/14/2018	No class	University closed due to weather.			
8 Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018	
9 Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018	
10 Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion			
11 Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018	
12 Wed, 9/26/2018	Chap. 3 and 6	Liouville theorem	#10	10/3/2018	
Fri, 9/28/2018	Chap. 3 and 6	Canonical transformations			
14 Mon, 10/1/2018	Chap. 4	Small oscillations about equilibrium	#11	10/5/2018	
15 Wed, 10/3/2018	Chap. 4	Normal modes of vibration			
16 Fri, 10/5/2018	Chap. 1-4, 6	Review			

The screenshot shows the Wake Forest University Physics website. The header features the WFU Physics logo and the text "Wake Forest College & Graduate School of Arts and Sciences". Below the header is a navigation bar with links for "WFU Physics", "People", "Events and News", "Undergraduate", "Graduate", "Research", and "Resources". To the right of the navigation bar are social media icons for Facebook, Twitter, and Google+. A search icon is also present. A red circle highlights the "Events" section on the right side of the page. The main content area displays a photograph of two students wearing safety goggles, focused on a green laser beam. The background shows laboratory equipment, including a computer monitor displaying "anta-Ray".

WFU Physics People Events and News Undergraduate Graduate Research Resources

Browse: Home

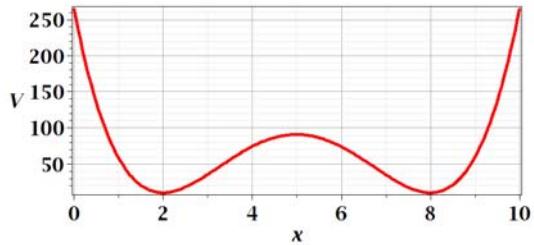
Events

Colloquium: "Adaptive Modeling of Vegetation Green-up and Die-off Responses to Extreme Weather" – Wednesday, October 3, 2018, at 4:00 PM
(Professor Lauren Lowman, Assistant Professor, Department of Earth and Atmospheric Sciences, University George P. Williams, Lecture Hall, (OHN 101) Wednesday, October 3, 2018, at 4:00 PM There will be a ...)

Colloquium: "Reading, Truth, and Objectivity: Reflections on Steven Weinberg's 'Against Philosophy'" – Wednesday, October 10, 2018, at 4:00 PM
(Professor Gordon Brittan, Montana State University George P. Williams, Jr. Lecture Hall, (OHN 101) Wednesday, October 10, 2018, at 4:00 PM)

Motivation for studying small oscillations – many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} = V(x_{eq}) + \frac{1}{2}k(x - x_{eq})^2$$



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Equations of motion for a single oscillator:

Let $k \equiv m\omega^2$

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -m\omega^2 x$$

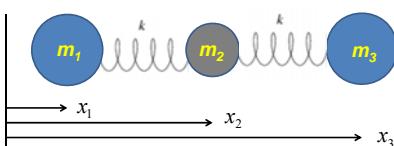
$$x(t) = A \sin(\omega t + \varphi)$$

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Example – linear molecule



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - \ell_{12})^2 - \frac{1}{2}k(x_3 - x_2 - \ell_{23})^2$$

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$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - \ell_{12})^2 - \frac{1}{2}k(x_3 - x_2 - \ell_{23})^2$$

$$\text{Let: } x_1 \rightarrow x_1 - x_1^0 \quad x_2 \rightarrow x_2 - x_1^0 - \ell_{12} \quad x_3 \rightarrow x_3 - x_1^0 - \ell_{12} - \ell_{23}$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2$$

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

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Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$

$$-\omega_a^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k \left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha \right)$$

$$-\phi^2 m_2 X_2^\alpha = -k(X_2^\alpha - X_3^\alpha)$$

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Coupled linear equations:

$$-\omega_g^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k \left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha \right)$$

$$-\omega^2 m_2 X_2^\alpha = -k(X_2^\alpha - X_2^\alpha)$$

Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

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Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i} X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11} - \omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} - \omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} - \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

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Digression:

Eigenvalue properties of matrices $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix : $H_{ij} = H^*_{ji}$

Theorem for Hermitian matrices :

λ_α have real values and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

Unitary matrix : $\mathbf{U}\mathbf{U}^H = \mathbf{I}$

$|\lambda_\alpha| = 1$ and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

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Digression on matrices -- continued

Eigenvalues of a matrix are "invariant" under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}(\mathbf{S}^{-1}\mathbf{y}'_\alpha) = \lambda'_\alpha (\mathbf{S}^{-1}\mathbf{y}'_\alpha)$$

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Example of transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_a^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

Let $S = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}$; $SMS^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$

Let $Y = SX$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_a^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

$$\text{where } \kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$$

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In our case :

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_a^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$ (CO_2)

$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_a^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

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Eigenvalues and eigenvectors :

$$\omega_1^2 = 0 \quad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

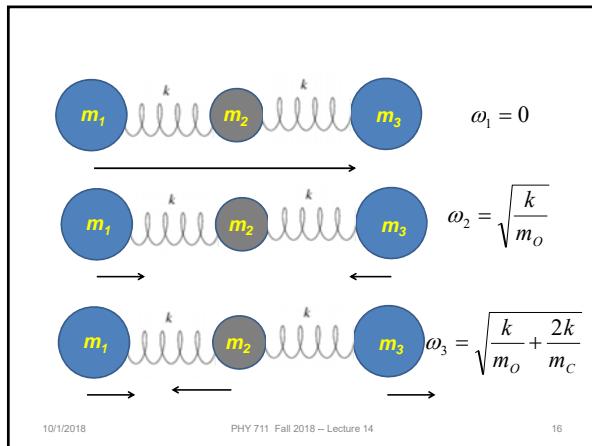
$$\omega_2^2 = \frac{k}{m_O} \quad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \quad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

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General solution :

$$x_i(t) = \Re \left(\sum_{\alpha} C^{\alpha} X_i^{\alpha} e^{-i\omega_{\alpha} t} \right)$$

For example, normal mode amplitudes C^α can be determined from initial conditions

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Additional digression on matrix properties Singular value decomposition

It is possible to factor any real matrix \mathbf{A} into unitary matrices \mathbf{V} and \mathbf{U} together with positive diagonal matrix Σ :

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}$$

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Singular value decomposition -- continued

Consider using SVD to solve a singular linear algebra problem $\mathbf{AX} = \mathbf{B}$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$$

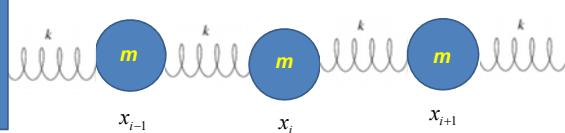
$$\mathbf{X} = \sum_{i \text{ for } \sigma_i > \varepsilon} \mathbf{v}_i \frac{\langle \mathbf{u}_i^H | \mathbf{B} \rangle}{\sigma_i}$$

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Consider an extended system of masses and springs:



Note : each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

Note : In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

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$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k(x_2 - 2x_1)$$

$$m\ddot{x}_2 = k(x_3 - 2x_2 + x_1)$$

.....

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

.....

$$m\ddot{x}_N = k(x_{N-1} - 2x_N)$$

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Matrix formulation --

$$\text{Assume } x_i(t) = X_i e^{-i\omega t}$$

$$\frac{m}{k} \omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem --

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This example also has an algebraic solution --

From Euler - Lagrange equations :

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try: } x_j(t) = A e^{-i\omega t + iqa j}$$

$$-\omega^2 A e^{-i\omega t + iqa j} = \frac{k}{m} (e^{iqa} - 2 + e^{-iqa}) A e^{-i\omega t + iqa j}$$

$$-\omega^2 = \frac{k}{m} (2 \cos(qa) - 2)$$

$$\Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

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From Euler-Lagrange equations -- continued:

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try: } x_j(t) = A e^{-i\omega t + iqa j} \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

$$\text{Note that: } x_j(t) = B e^{-i\omega t - iqa j} \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

General solution:

$$x_j(t) = \Re(A e^{-i\omega t + iqa j} + B e^{-i\omega t - iqa j})$$

Impose boundary conditions:

$$x_0(t) = \Re(A e^{-i\omega t + iqa j} + B e^{-i\omega t - iqa j}) = 0$$

$$x_{N+1}(t) = \Re(A e^{-i\omega t + iqa(N+1)} + B e^{-i\omega t - iqa(N+1)}) = 0$$

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Impose boundary conditions -- continued:

$$x_0(t) = \Re(A e^{-i\omega t} + B e^{i\omega t}) = 0$$

$$x_{N+1}(t) = \Re \left(A e^{-i\omega t + iqa(N+1)} + B e^{-i\omega t - iqa(N+1)} \right) = 0$$

$$\Rightarrow B = -A$$

$$x_{N+1}(t) = \Re \left(A e^{-i\omega t} \left(e^{iqa(N+1)} - e^{-iqa(N+1)} \right) \right) = 0$$

$$\Rightarrow \sin(qa(N+1)) = 0$$

$$\Rightarrow qa(N+1) = v\pi \quad \text{where } v = 0, 1, 2, \dots$$

$$qa = \frac{v\pi}{N+1}$$

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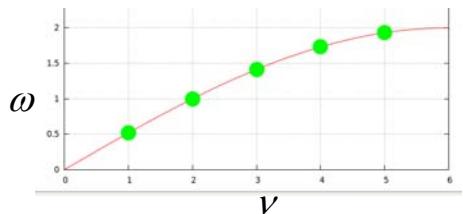
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Summary of results:

$$\Rightarrow \omega_v^2 = \frac{4k}{m} \sin^2 \left(\frac{v\pi}{2(N+1)} \right) \quad x_n = \Re \left(2iA \sin \left(\frac{v\pi n}{N+1} \right) \right)$$

$$\nu = 0, 1, \dots N$$

$$n = 1, 2, \dots, N$$



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