PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 18:

Continue reading Chapter 7

1. More comments on non-linear equation example
2. The wave equation
3. Sturm-Liouville equation
4. Green's function solution methods

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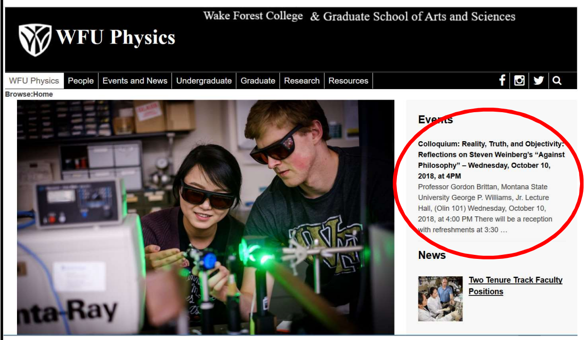

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	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018
12	Wed, 9/26/2018	Chap. 3 and 6	Liouville theorem	#10	10/3/2018
13	Fri, 9/28/2018	Chap. 3 and 6	Canonical transformations		
14	Mon, 10/1/2018	Chap. 4	Small oscillations about equilibrium	#11	10/5/2018
15	Wed, 10/3/2018	Chap. 4	Normal modes of vibration		
16	Fri, 10/5/2018	Chap. 1-4, 6	Review		
17	Mon, 10/8/2018	Chap. 7	Strings		
18	Wed, 10/10/2018	Chap. 7	Wave equation		
	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		

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Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

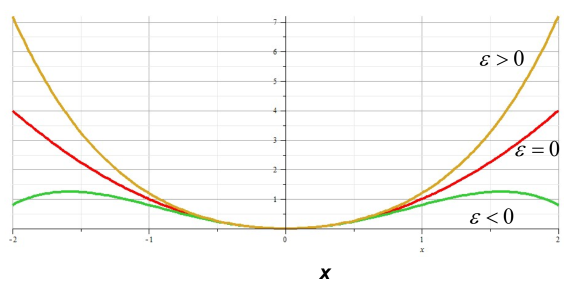
$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2\left(x + \varepsilon x^3\right)$$

Superposition - - no longer applies

$$V(x) \approx \frac{1}{2}m\omega^2\left(x^2 + \varepsilon x^4\right)$$



Non - linear example - - continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler - Lagrange equations :

zero order : $\ddot{x}_0 + \omega^2 x_0 = 0$

first order : $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Non-linear example -- continued

$$\ddot{x}+\omega^2\left(x+\varepsilon x^3\right)=0$$

Initial conditions:

$$x(0)=X_0 \qquad \dot{x}(0)=0$$

Perturbation expansion:

$$x(t)=x_0(t)+\varepsilon x_1(t)+\ldots$$

Euler-Lagrange equations:

zero order: $\ddot{x}_0+\omega^2 x_0=0 \qquad \Rightarrow x_0(t)=X_0\cos(\omega t)$

first order: $\ddot{x}_1+\omega^2 x_1+\omega^2 x_0^3=0$

$$\Rightarrow \ddot{x}_1(t)+\omega^2 x_1(t)=-\omega^2 X_0^3\cos^3(\omega t)=-\frac{\omega^2 X_0^3}{4}\left(3\cos(\omega t)+\cos(3\omega t)\right)$$

$$\Rightarrow x_1(t)=-\frac{X_0^3}{8}\left\{3\omega t\,\sin(\omega t)+\frac{1}{4}\left[\cos(\omega t)-\cos(3\omega t)\right]\right\}$$

$$x(t)=X_0\cos(\omega t)-\varepsilon\frac{X_0^3}{8}\left\{3\omega t\,\sin(\omega t)+\frac{1}{4}\left[\cos(\omega t)-\cos(3\omega t)\right]\right\}+O(\varepsilon^2)$$

Blows up at large *t*

Some details:

$$\ddot{x}+\omega^2\left(x+\varepsilon x^3\right)=0$$

Initial conditions:

$$x(0)=X_0 \qquad \dot{x}(0)=0$$

Perturbation expansion:

Let: $\tau\equiv\omega(\epsilon)t$: where $\omega(\epsilon)=\omega+\epsilon\omega_1+\ldots\ldots \qquad u(\tau)=u_0(\tau)+\varepsilon u_1(\tau)+\ldots$

Differential equation: $\left(\omega(\epsilon)\right)^2\ddot{u}+\omega^2\left(u+\epsilon u^3\right)=0$

Zero order term: $\omega^2\frac{d^2u_0}{d\tau^2}+\omega^2 u_0=0 \qquad \Rightarrow u_0(t)=X_0\cos(\omega t)$

First order term: $\omega^2\frac{d^2u_1}{d\tau^2}+\omega^2 u_1=-2\omega\omega_1\frac{d^2u_0}{d\tau^2}-\omega^2u_0^3=2\omega\omega_1X_0\cos(\omega t)-\omega^2X_0^3\cos^3(\omega t)$

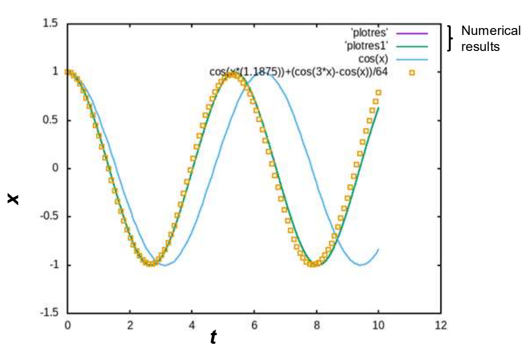
$$=\left(2\omega\omega_1X_0-\frac{3\omega^2X_0^3}{4}\right)\cos(\omega t)-\frac{\omega^2X_0^3}{4}\cos(3\omega t)$$

Let $\omega_1=\frac{3\omega X_0^2}{8} \qquad \Rightarrow \qquad \frac{d^2u_1}{d\tau^2}+u_1=-\frac{X_0^3}{4}\cos(3\tau)$

$$u_1(\tau)=\frac{X_0^3}{32}\cos(3\tau)+A\cos(\tau)=\frac{X_0^3}{32}\cos(3\tau)-\frac{X_0^3}{32}\cos(\tau)$$

$$x(t)=X_0\cos\left(\omega\left(1+\frac{3\epsilon X_0^2}{8}\right)t\right)+\epsilon\frac{X_0^3}{32}\left(\cos(3\omega t)-\cos(\omega t)\right)+O(\epsilon^2)$$

Result for ω=1, ε=0.5



Back to linear equations --

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu}{\partial x}\right) + v(x)\mu = 0$$

Separating time and spatial variables:

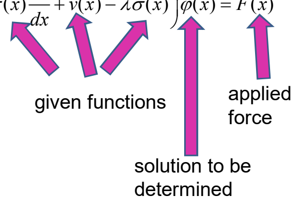
$$\mu(x,t) = \rho(x)\cos(\omega t + \phi)$$

Sturm-Liouville equation for spatial function:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\rho}{dx}\right) + v(x)\rho = \omega^2\sigma(x)\rho$$

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)\varphi(x) = F(x)$



Homogenous problem: *F(x)=0*

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)\varphi(x) = 0$$

Bessel functions:

$$\tau(x) = -x \quad v(x) = x \quad \sigma(x) = \frac{1}{x} \quad \lambda = \nu^2 \quad \varphi(x) = J_\nu(x)$$

Legendre functions:

$$\tau(x) = -\left(1 - x^2\right) \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = l(l+1) \quad \varphi(x) = P_l(x)$$

Fourier functions:

$$\tau(x) = 1 \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = n^2\pi^2 \quad \varphi(x) = \sin(n\pi x)$$

Solution methods of Sturm-Liouville equations
(assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx.$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Comment on “completeness”

It can be shown that for any reasonable function *h(x)*, defined within the interval *a < x <b*, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x)\approx\sum_n C_nf_n(x),$$

where $C_n=\frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval *a < x <b,* as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x').$$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle\tilde{h}|S|\tilde{h}\rangle}{\langle\tilde{h}|\sigma|\tilde{h}\rangle}, \qquad S(x)\equiv -\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x)=\sum_n C_nf_n(x),$ where the coefficients C_n can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 \; N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 \; N_n,$$

Therefore $\dfrac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \dfrac{\sum_n |C_n|^2 \; N_n \lambda_n}{\sum_n |C_n|^2 \; N_n} \geq \lambda_0.$

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

Example: $-\dfrac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x)$ with $f_n(0) = f_n(a) = 0$

trial function $f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \dfrac{\pi^2}{a^2} = \dfrac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\dfrac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \dfrac{10}{a^2}$

Green's function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) G_\lambda(x,x') = \delta\left(x - x'\right)$$

Completeness of eigenfunctions:

Recall:

$$\sigma\left(x\right)\sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta\left(x - x'\right)$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) G_\lambda(x,x') = \sigma\left(x\right)\sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example : $\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method :

$$G(x,x')=\frac{1}{W}g_a(x_<)g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x);\quad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\sin(L)$$

$$\phi(x)=\phi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x\sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L\sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\phi(x)=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

General method of constructing Green's functions using homogeneous solution

Green's function:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

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For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon} dx\delta(x-x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

$$\Rightarrow W=\tau(x')\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'}=0$.

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

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$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u}G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x}g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u}g_b(x')F(x')dx'$$

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