PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 19:

Read Chapter 7 & Appendices A-D

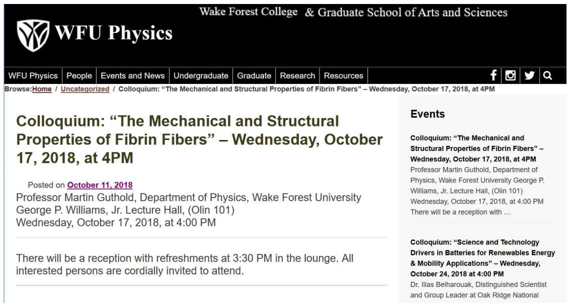
Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Sturm-Liouville equations**
2. **Eigenvalues; orthogonal function expansions**
3. **Green's functions methods**
4. **Laplace transformation**
5. **Contour integration methods**

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018
12	Wed, 9/26/2018	Chap. 3 and 6	Liouville theorem	#10	10/3/2018
13	Fri, 9/28/2018	Chap. 3 and 6	Canonical transformations		
14	Mon, 10/1/2018	Chap. 4	Small oscillations about equilibrium	#11	10/5/2018
15	Wed, 10/3/2018	Chap. 4	Normal modes of vibration		
16	Fri, 10/5/2018	Chap. 1-4, 6	Review		
17	Mon, 10/8/2018	Chap. 7	Strings		
18	Wed, 10/10/2018	Chap. 7	Wave equation		
	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		

Remember:

Mid term exam due tomorrow (Tuesday) --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Browse: Home / Uncategorized / Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM

Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM

Posted on October 11, 2018

Professor Martin Guthold, Department of Physics, Wake Forest University
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, October 17, 2018, at 4:00 PM

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

Events

Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM
Professor Martin Guthold, Department of Physics, Wake Forest University George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, October 17, 2018, at 4:00 PM There will be a reception with …

Colloquium: "Science and Technology Drivers in Batteries for Renewables Energy & Mobility Applications" – Wednesday, October 24, 2018 at 4:00 PM
Dr. Ilias Belharouak, Distinguished Scientist and Group Leader at Oak Ridge National

Eigenvalues and eigenfunctions of Sturm-Liouville equations

In the domain $a \le x \le b$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x),$ where the coefficients C_n can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 \; N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 \; N_n,$$

$$\text{Therefore} \quad \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \frac{\sum_n |C_n|^2 \; N_n \lambda_n}{\sum_n |C_n|^2 \; N_n} \geq \lambda_0.$$

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

$$\text{Example:} \quad -\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(0) = f_n(a) = 0$$

$$\text{trial function} \quad f_{\text{trial}}(x) = x(x-a)$$

$$\text{Exact value of } \lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$$

$$\text{Raleigh-Ritz estimate:} \quad \frac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \frac{10}{a^2}$$

Rayleigh-Ritz method of estimating the lowest eigenvalue

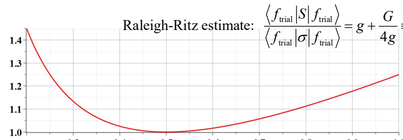
$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

$$\text{Example:} \quad -\frac{d^2 f_n(x)}{dx^2} + Gx^2 f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(-\infty) = f_n(\infty) = 0$$

$$\text{trial function} \quad f_{\text{trial}}(x) = e^{-gx^2}$$

$$\text{Raleigh-Ritz estimate:} \quad \frac{\left\langle f_{\text{trial}} \middle| S \middle| f_{\text{trial}} \right\rangle}{\left\langle f_{\text{trial}} \middle| \sigma \middle| f_{\text{trial}} \right\rangle} = g + \frac{G}{4g} \equiv \lambda_{\text{trial}}(g)$$

$$\frac{\lambda_{\text{trial}}(g)}{\sqrt{G}}$$



$$g / \sqrt{G}$$

$$g_0 = \frac{1}{2}\sqrt{G} \qquad \lambda_{\text{trial}}(g_0) = \sqrt{G}$$

Note that for differential equation of the Schoedinger equation of the harmonic oscillator:

$$\sqrt{G} = \frac{m\omega}{\hbar} \qquad \lambda_{\text{trial}} = \frac{2m}{\hbar^2} E_0 \qquad \Rightarrow E_0 = \frac{\hbar\omega}{2}$$

Recap -- Rayleigh-Ritz method of estimating the lowest eigenvalue

Example from Schroedinger equation for one-dimensional harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2 f_n(x)}{dx^2}+\frac{1}{2}m\omega^2 x^2 f_n(x)=E_n f_n(x) \qquad \text{with } f_n(-\infty)=f_n(\infty)=0$$

Trial function $f_{\text{trial}}(x)=e^{-gx^2}$

Raleigh-Ritz estimate: $\frac{\langle f_{\text{trial}}|S|f_{\text{trial}}\rangle}{\langle f_{\text{trial}}|\sigma|f_{\text{trial}}\rangle}=\frac{\hbar^2}{2m}\left(g+\frac{m^2\omega^2/\hbar^2}{4g}\right)\equiv E_{\text{trial}}(g)$

$g_0=\frac{m\omega}{\hbar} \qquad E_{\text{trial}}(g_0)=\frac{1}{2}\hbar\omega$ ⇐ Exact answer

Comment on "completeness" of set of eigenfunctions

It can be shown that for any reasonable function *h(x)*, defined within the interval *a* < *x* <*b*, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x)\approx\sum_n C_n f_n(x),$$

where $C_n=\frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval *a* < *x* <*b,* as summarized by the statement:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x').$$

Some details – suggested that: $h(x)\approx\sum_n C_n f_n(x),$

where $C_n=\frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

Minimize:

$$\chi(\{C_n\})=\int_a^b dx\ \sigma(x)\left(h(x)-\sum_n C_n f_n(x)\right)^2$$

Necessary condition for minimum:

$$\frac{d\chi}{dC_n}=0 \qquad \int_a^b dx\ 2\sigma(x)\left(h(x)-\sum_m C_m f_m(x)\right)f_n(x)=0$$

Note that: $\int_a^b \sigma(x')f_m(x')f_n(x')dx'=N_n\delta_{mn}$

$$\Rightarrow C_n=\frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'$$

Green's function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Among other things, this is useful for solving inhomogeneous equations of the type:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\psi(x)=F(x)$$

where $F(x),\tau(x),v(x),\lambda,$ and $\sigma(x)$ are known, and $\psi(x)$ is to be determined according to:

$$\psi(x)=\int_a^b dx' G(x,x')F(x')$$

Green's function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Completeness of eigenfunctions:

Recall:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem for $0\le x\le L$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\ \ \sigma(x)=1;\ \ v(x)=0;\ \ a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues:

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

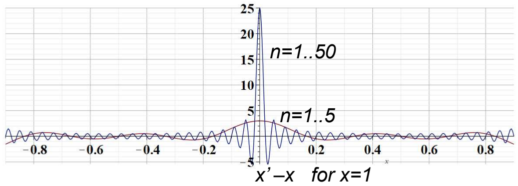
Completeness of eigenfunctions:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example: $\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$

Example: $\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$

For *L=2*



Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method:

$$G(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x); \qquad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\sin(L)$$

$$\varphi(x)=\varphi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x\sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L\sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\varphi(x)=\varphi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

(after some algebra)

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

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For $\epsilon\to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon}dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon}dx\delta(x-x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon}dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

$$\Rightarrow W=\tau(x')\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'}=0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

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$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u}G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x}g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u}g_b(x')F(x')dx'$$

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