PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 20:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. Orthogonal function expansions
2. Fourier series
3. Fourier transforms
4. Fast Fourier transforms

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Browse:Home / Uncategorized / Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM

Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM

Posted on October 11, 2018

Professor Martin Guthold, Department of Physics, Wake Forest University
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, October 17, 2018, at 4:00 PM

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

Events

Colloquium: "The Mechanical and Structural Properties of Fibrin Fibers" – Wednesday, October 17, 2018, at 4PM
Professor Martin Guthold, Department of Physics, Wake Forest University George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, October 17, 2018, at 4:00 PM There will be a reception with …

Colloquium: "Science and Technology Drivers in Batteries for Renewables Energy & Mobility Applications" – Wednesday, October 24, 2018 at 4:00 PM
Dr. Ilias Belharouak, Distinguished Scientist and Group Leader at Oak Ridge National

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		
11	Mon, 9/24/2018	Chap. 3 and 6	Hamiltonian formalism	#9	9/28/2018
12	Wed, 9/26/2018	Chap. 3 and 6	Liouville theorem	#10	10/3/2018
13	Fri, 9/28/2018	Chap. 3 and 6	Canonical transformations		
14	Mon, 10/1/2018	Chap. 4	Small oscillations about equilibrium	#11	10/5/2018
15	Wed, 10/3/2018	Chap. 4	Normal modes of vibration		
16	Fri, 10/5/2018	Chap. 1-4, 6	Review		
17	Mon, 10/8/2018	Chap. 7	Strings		
18	Wed, 10/10/2018	Chap. 7	Wave equation		
	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		
20	Wed, 10/17/2018	Chap. 7	Fourier Transforms	#12	10/22/2018

Eigenvalues and eigenfunctions of Sturm-Liouville equations

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2\,dx.$

Special case: $\tau(x)=1=\sigma(x) \quad v(x)=0$

$$-\frac{d^2}{dx^2}f_n(x)=\lambda_n f_n(x) \quad \text{for } 0 \le x \le a, \quad \text{with } f_n(0)=f_n(a)=0$$

Special case: $\tau(x)=1=\sigma(x) \quad v(x)=0$

$$-\frac{d^2}{dx^2}f_n(x)=\lambda_n f_n(x) \quad \text{for } 0 \le x \le a, \quad \text{with } f_n(0)=f_n(a)=0$$

$$f_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad \lambda_n=\left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x)=\sum_{n=1}^{\infty}A_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

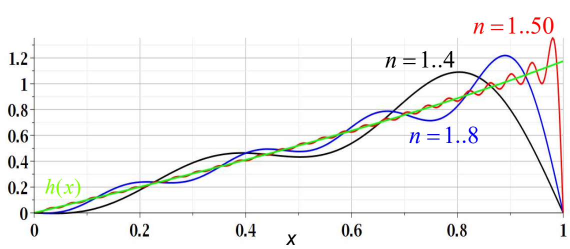
$$A_n=\sqrt{\frac{2}{a}}\int_0^a dx'\ h(x')\ \sin\left(\frac{n\pi x'}{a}\right)$$

*Note that if $h(x)$ does not vanish at $x=0$ and $x=a$, the more general expression applies: $h(x)=\sum_{n=1}^{\infty}A_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)+\sum_{n=0}^{\infty}B_n\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)$

(with some restrictions).

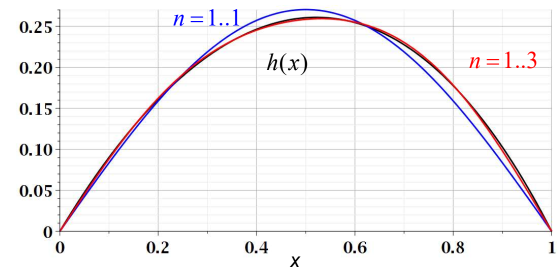
Example

$$h(x)=\sinh(x)\approx 2\pi\sinh(1)\left(\frac{\sin(\pi x)}{\pi^2+1}-\frac{2\sin(2\pi x)}{4\pi^2+1}+\ldots-(-1)^n n\frac{\sin(n\pi x)}{n^2\pi^2+1}+\ldots\right)$$



Example

$$h(x) = (1-x)\sinh(x) \approx 4\pi\left(\frac{\sin(\pi x)(1+\sinh(1))}{(\pi^2+1)^2} + \frac{2\sin(2\pi x)(1-\sinh(1))}{(4\pi^2+1)^2} + \ldots + n\frac{\sin(n\pi x)\left(1-(-1)^n\sinh(1)\right)}{(n^2\pi^2+1)^2} + \ldots\right)$$



Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx'\ h(x')\ \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is piecewise continuous.

Generalization to infinite range

Examples in time domain --

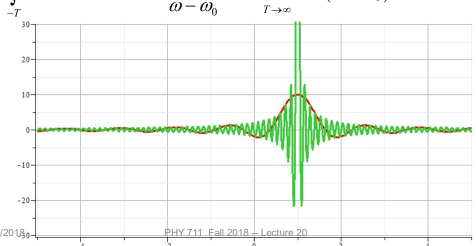
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\ e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\ e^{-i(\omega-\omega_0)t} = \frac{2\sin\left[(\omega-\omega_0)T\right]}{\omega-\omega_0} \underset{T\to\infty}{\approx} 2\pi\delta(\omega-\omega_0)$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t)=\int_{-\infty}^{\infty}d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t)=\int_{-\infty}^{\infty}d\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\, f(t')\, e^{i\omega t'}\right)e^{-i\omega t}$$

$$f(t)=\int_{-\infty}^{\infty}dt'\, f(t')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t'-t)}\right)=\int_{-\infty}^{\infty}dt'\, f(t')\,\delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty}dt\,\left(f(t)\right)^{*} f(t)=2\pi\int_{-\infty}^{\infty}d\omega\,\left(F(\omega)\right)^{*}F(\omega)$$

Check:

$$\int_{-\infty}^{\infty}dt\,\left(f(t)\right)^{*} f(t)=\int_{-\infty}^{\infty}dt\left(\left(\int_{-\infty}^{\infty}d\omega F(\omega)e^{i\omega t}\right)^{*}\int_{-\infty}^{\infty}d\omega' F(\omega')e^{i\omega' t}\right)$$

$$=\int_{-\infty}^{\infty}d\omega F^{*}(\omega)\int_{-\infty}^{\infty}d\omega' F(\omega')\int_{-\infty}^{\infty}dt\, e^{i(\omega'-\omega)t}$$

$$=\int_{-\infty}^{\infty}d\omega F^{*}(\omega)\int_{-\infty}^{\infty}d\omega' F(\omega')\,2\pi\delta(\omega'-\omega)$$

$$=2\pi\int_{-\infty}^{\infty}d\omega F^{*}(\omega)F(\omega)$$

Use of Fourier transforms to solve wave equation

Wave equation: $\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$

Suppose $u(x,t)=e^{-i\omega t}\widetilde{F}(x,\omega)$ where $\widetilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2\widetilde{F}(x,\omega)}{\partial x^2}=-\frac{\omega^2}{c^2}\widetilde{F}(x,\omega)\equiv -k^2\widetilde{F}(x,\omega)$$

More generally:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega F(x,\omega)e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0\leq x\leq L$

with $\widetilde{F}(0,\omega)=0$ and $\widetilde{F}(L,\omega)=0$

For $n=1,2,3\cdots$

$$\widetilde{F}_n(x,\omega)=\sin\left(\frac{n\pi x}{L}\right)\qquad k\rightarrow k_n=\frac{n\pi}{L}\equiv\frac{\omega_n}{c}$$

$$u(x,t)=e^{-i\omega_n t}\sin(k_n x)=e^{-i\omega_n t}\frac{\left(e^{ik_n x}-e^{-ik_n x}\right)}{2i}=\frac{\left(e^{ik_n(x-ct)}-e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t} \widetilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \widetilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \widetilde{F}(x,\omega_n) \equiv -k_n^2 \widetilde{F}(x,\omega_n)$$

For $\widetilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\quad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{C_n}{2i} e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i}\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Fourier transform for a time periodic function:

Suppose $f(t+nT) = f(t)$ for any integer n

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega(t+nT)}\right)$$

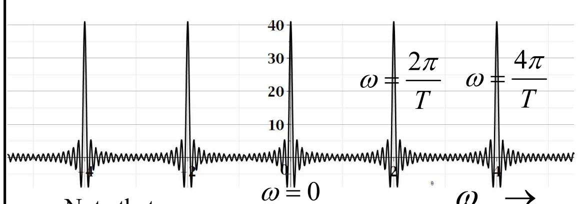
Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Details:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \lim_{N\to\infty} \sum_{n=-N}^{N} e^{in\omega T} = \lim_{N\to\infty} \frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$

$$\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$



Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_{\nu}\delta(\omega T-\nu\Omega T) = \frac{2\pi}{T}\sum_{\nu}\delta(\omega-\nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where} \quad \Omega\equiv\frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t} = \sum_{\nu=-\infty}^{\infty}\Omega\delta(\omega-\nu\Omega)\left(\int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t}\right)$$

Thus, for a time periodic function

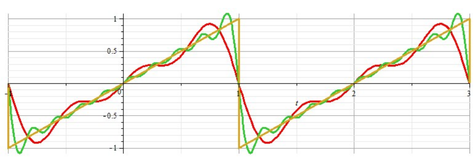
$$f(t) = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$$

Example:

Suppose: $f(t) = \begin{cases} \frac{t-nT}{T} & \text{for } (n-1)T \le t \le (n+1)T; \; n = 0,2,4,6\ldots \\ 0 & \text{otherwise} \end{cases}$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T \le t \le T$.

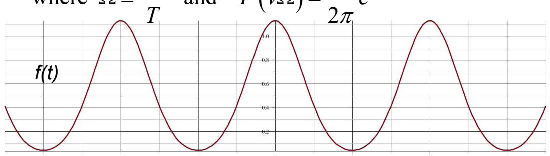
$$\bar{F}(\nu\Omega) = \frac{1}{2T}i\int_{-T}^{T}\frac{t}{T}\sin\left(\frac{\nu 2\pi t}{2T}\right)dt \qquad f(t) = \sum_{\nu=1}^{\infty}2\left|\bar{F}(\nu\Omega)\right|\sin\left(\frac{\nu 2\pi t}{2T}\right)$$

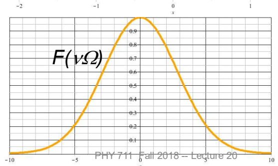


Example:

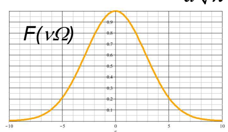
Suppose: $f(t) = \frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$

where $\Omega\equiv\frac{2\pi}{T}$ and $F(\nu\Omega) = \frac{1}{2\pi}e^{-a^2\nu^2\Omega^2/4}$





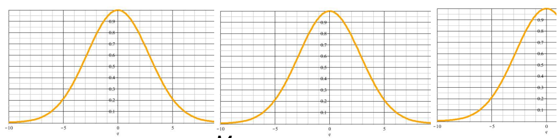
Continued: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$



$\Omega = \frac{2\pi}{T}$

Note:

$f(t) \approx \sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i\nu\Omega t}$



ν=-M

ν=M

For $t=\frac{m}{2M+1}T: \Rightarrow f\left(\frac{mT}{2M+1}\right)=\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i2\pi\nu m/(2M+1)}$

Thus, for a periodic function

$f(t)=\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$

Now suppose that the transformed function is bounded;

$|F(\nu\Omega)| \leq \varepsilon \quad \text{for} \quad |\nu| \geq N$

Define a periodic transform function function

$\widetilde{F}(\nu\Omega) \equiv \widetilde{F}(\nu\Omega + \nu'((2N+1)\Omega))$

Effect on time domain :

$f(t)=\sum_{\nu=-\infty}^{\infty} \widetilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N} \widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$

Doubly periodic functions

$$t \rightarrow \frac{\mu T}{2N+1}$$

$$\widetilde{f}_{\mu} = \frac{1}{2N+1}\sum_{\nu=-N}^{N} \widetilde{F}_{\nu} e^{-i2\pi\nu\mu/(2N+1)}$$

$$\widetilde{F}_{\nu} = \sum_{\mu=-N}^{N} \widetilde{f}_{\mu} e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1 \rightarrow M$$

$$\tilde{f}_{\mu} = \frac{1}{M}\sum_{\nu=0}^{M-1} \tilde{F}_{\nu} e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_{\nu} = \sum_{\mu=0}^{M} \tilde{f}_{\mu} e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)

http://www.fftw.org/

FFTW

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Introduction

FFTW is a C subroutine library for computing the discrete Fourier transform (DFT) in one or more dimensions, of arbitrary input size, and of both real and complex data (as well as of even/odd data, i.e. the discrete cosine/sine transforms or DCT/DST). We believe that FFTW, which is free software, should become the FFT library of choice for most applications.

The latest official release of FFTW is version **3.3.8**, available from our download page. Version 3.3 introduced support for the AVX x86 extensions, a distributed-memory implementation on top of MPI, and a Fortran 2003 API. Version 3.3.1 introduced support for the ARM Neon extensions. See the release notes for more information.

The FFTW package was developed at MIT by Matteo Frigo and Steven G. Johnson.

Our benchmarks, performed on on a variety of platforms, show that FFTW's performance is typically superior to that of other publicly available FFT software, and is even competitive with vendor-tuned codes. In contrast to vendor-tuned codes, however, FFTW's performance is *portable*: the same program will perform well on most architectures without modification. Hence the name, "FFTW," which stands for the somewhat whimsical title of "**Fastest Fourier Transform in the West**."