PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103
Plan for Lecture 25:

Motions of elastic membranes (Chap. 8)

1. **Review of standing waves on a string**
2. **Standing waves on a two dimensional membrane.**
3. **Boundary value problems**

Colloquium: "Energy Landscapes for Protein Folding and Misfolding" Dr. Peter Wolynes, Wednesday, October 31, 2018, at 4:00 PM

Dr. Peter G. Wolynes, Bullard-Welch Foundation Professor of Science;
Professor of Chemistry, MSNE, and Physics and Astronomy at Rice University,
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, October 31, 2018, at 4:00 PM
(Colloquium sponsored jointly with WFU Dept. of Chemistry)

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

18	Wed, 10/10/2018	Chap. 7	Wave equation		
	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		
20	Wed, 10/17/2018	Chap. 7	Fourier Transforms	#12	10/22/2018
21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018				
27	Fri, 11/02/2018				
28	Mon, 11/05/2018				
29	Wed, 11/07/2018				
30	Fri, 11/09/2018				
31	Mon, 11/12/2018				
32	Wed, 11/14/2018				
33	Fri, 11/16/2018				
34	Mon, 11/19/2018				
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018				

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

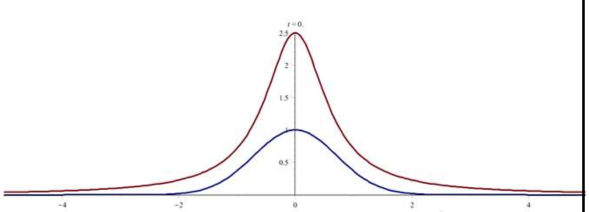
For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \frac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

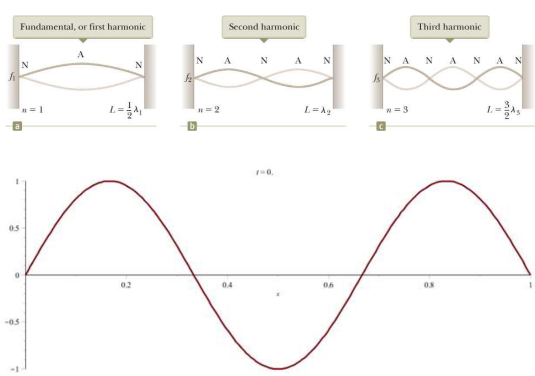
with $\mu(0,t) = \mu(L,t) = 0.$

Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\dfrac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0 \qquad k = \dfrac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$



Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

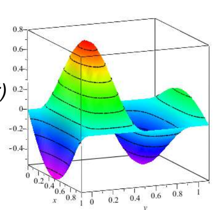
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

$u(x,y,t)$



Lagrangian density: $\mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)$

$$L=\int \mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)dxdy$$

Hamilton's principle:

$$\delta\int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u}-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)}-\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)}-\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)}=0$$

Lagrangian density for elastic membrane with constant σ and τ:

$$\mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)=\frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u}-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)}-\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)}-\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)}=0$$

$$\frac{\partial^2 u}{\partial t^2}-c^2\nabla^2 u=0 \qquad \text{where } c^2=\frac{\tau}{\sigma}$$

Two-dimensional wave equation:

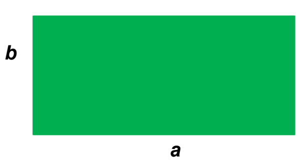
$$\frac{\partial^2 u}{\partial t^2}-c^2\nabla^2 u=0 \qquad \text{where } c^2=\frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t)=\Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0 \qquad \text{where } k=\frac{\omega}{c}$$

Consider a rectangular boundary:



Clamped boundary conditions:

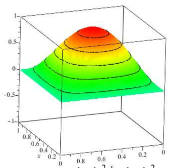
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

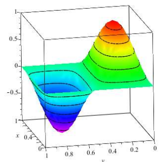
$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0$$

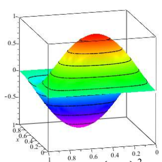
$$\text{where } k=\frac{\omega}{c}$$



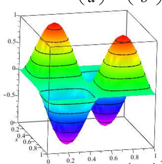
$k_{11}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



$k_{12}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$



$k_{21}^{\ 2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



$k_{22}^{\ 2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$

More general boundary conditions:

$\tau\nabla u\big|_b = \kappa u\big|_b$ represents bounded side constrained with spring

$\tau\nabla u\big|_b = 0$ represents "free" side

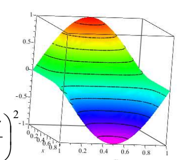
Mixed boundary conditions:

$$\rho(x,0) = \rho(x,b) = \frac{\partial\rho(0,y)}{\partial x} = \frac{\partial\rho(a,y)}{\partial x} = 0$$

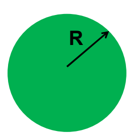
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$k_{11}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$\rho(R,\varphi) = 0$

$$\left(\nabla^2 + k^2\right)\rho(r,\varphi) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

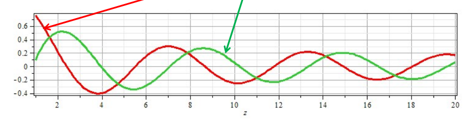
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

Asending series: $J_m(z)=\left(\frac{z}{2}\right)^m\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z)+J_{m+1}(z)=\frac{2m}{z}J_m(z)$

$$J_{m-1}(z)-J_{m+1}(z)=2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z)\xrightarrow[z>>1]{}\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{m\pi}{2}-\frac{\pi}{4}\right)$

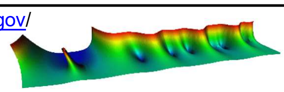
Zeros of Bessel functions $J_m(z_{mn})=0$

$m=0$: $z_{0n}=2.406,\ 5.520,\ 8.654,...$

$m=1$: $z_{1n}=3.832,\ 7.016,\ 10.173,...$

$m=2$: $z_{2n}=5.136,\ 8.417,\ 11.620,...$

http://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 DLMF Update; Version 1.0.9
2014-04-25 DLMF Update; Version 1.0.8; errata & improved MathML
2014-03-21 DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics
2013-08-16 Bille C. Carlson, DLMF Author, dies at age 89
More news

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right)J_n(z)$$
$$-\frac{\left(\tfrac{1}{2}z\right)^{n}}{\pi}\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!(n+k)!},$$

Some properties of Bessel functions -- continued

Note : It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

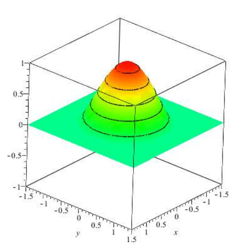
$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)r\,dr = \frac{R^2}{2}\left(J_{m+1}(z_{mn})\right)^2\delta_{nn'}$$

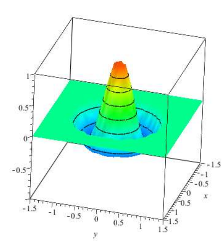
Returning to differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\ f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right) \qquad \rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$

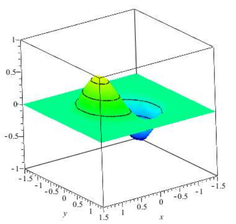


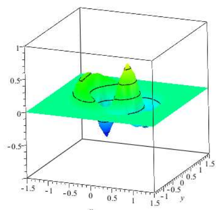


$$k_{01} = \frac{2.406}{R} \qquad k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi) = AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

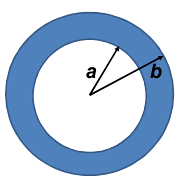
$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi) = AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$





$$k_{11} = \frac{3.832}{R} \qquad k_{12} = \frac{7.016}{R}$$

More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume : $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let : $\Phi(\varphi) = e^{im\varphi}$

Note : $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

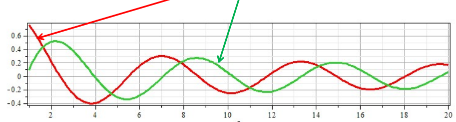
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right)f(r) = 0$$

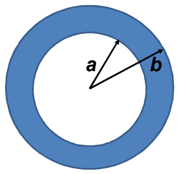
$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Normal modes of an annular membrane -- continued

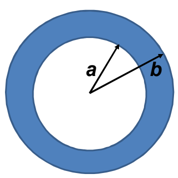


Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

Normal modes of an annular membrane -- continued



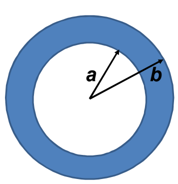
Boundary conditions:
$f(a)=0 \qquad f(b)=0$

$$AJ_m(ka)+BN_m(ka)=0$$
$$AJ_m(kb)+BN_m(kb)=0$$

$\Rightarrow$ 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)} \quad \text{(transcendental equation for } k\text{)}$$

Normal modes of an annular membrane -- continued



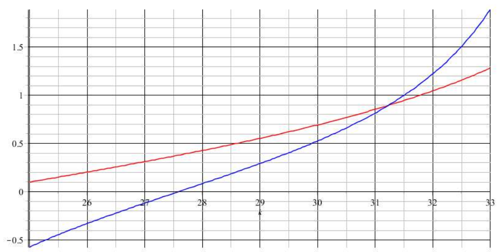
Boundary conditions:
$f(a)=0 \qquad f(b)=0$

$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)} \quad \text{-- in terms of solution } k_{mn}:$$

$$f(r)=A\left(J_m(k_{mn}r)-\frac{J_m(k_{mn}a)}{N_m(k_{mn}a)}N_m(k_{mn}r)\right)$$

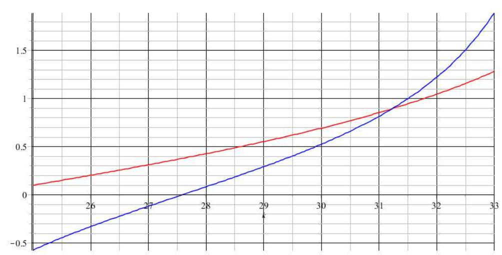
Analysis for *m=0* and a=0.1, b=0.2:

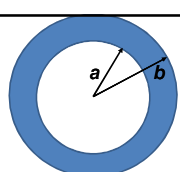
> $plot\left(\left\{\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)}, \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}\right\}, k = 25..33, color = [red, blue]\right);$



> $fsolve\left(\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)} = \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}, k, 30..33\right);$

31.23030920





$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)}N_m(k_{mn}r)\right) \qquad k_{01} = 31.230309$$