PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Chap. 8 in F & W: Summary of two-dimensional membrane analysis

Chap. 9 in F & W: Introduction to hydrodynamics
1. **Motivation for topic**
2. **Newton's laws for fluids**
3. **Conservation relations**

Today!

Colloquium: "Energy Landscapes for Protein Folding and Misfolding" Dr. Peter Wolynes, Wednesday, October 31, 2018, at 4:00 PM

Dr. Peter G. Wolynes, Bullard-Welch Foundation Professor of Science;
Professor of Chemistry, MSNE, and Physics and Astronomy at Rice University,
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, October 31, 2018, at 4:00 PM
(Colloquium sponsored jointly with WFU Dept. of Chemistry)

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

17	Mon, 10/8/2018	Chap. 7	Strings		
18	Wed, 10/10/2018	Chap. 7	Wave equation		
	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		
20	Wed, 10/17/2018	Chap. 7	Fourier Transforms	#12	10/22/2018
21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018	Chap. 9	Mechanics of three dimensional fluids		
27	Fri, 11/02/2018				
28	Mon, 11/05/2018				
29	Wed, 11/07/2018				
30	Fri, 11/09/2018				
31	Mon, 11/12/2018				
32	Wed, 11/14/2018				
33	Fri, 11/16/2018				
34	Mon, 11/19/2018				
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018				

Review --

Two - dimensional wave equation :

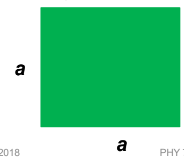
$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions :

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a square boundary:



Free boundary conditions:

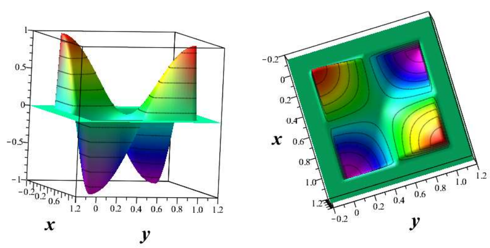
$$\frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = \frac{\partial \rho(x,a)}{\partial y} = \frac{\partial \rho(x,0)}{\partial y} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

For $n = m = 1$:

$$\rho_{11}(x,y) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$$



Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

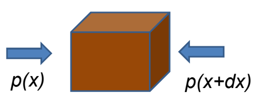
Velocity $\mathbf{v}(x,y,z,t)$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z) + p(x,y,z)\right)dydz$$

$$= \frac{\left(-p(x+dx,y,z) + p(x,y,z)\right)}{dx}dxdydz$$

$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p)dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

Detailed analysis of accelerati on term :

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}\right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla \Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force
3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C'(t)\right)$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0$$

Modified form; assuming $\frac{\partial\Phi}{\partial t}=0$

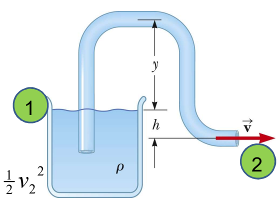
$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$

$p_1=p_2=p_{atm}$

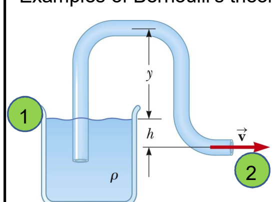
$U_1-U_2=gh$

$v_1\approx 0$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$



Examples of Bernoulli's theorem -- continued



$p_1=p_2=p_{atm}$

$U_1-U_2=gh$

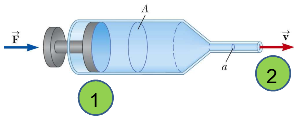
$v_1\approx 0$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

$$v_2\approx\sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$

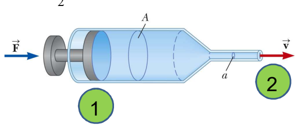
$$U_1 = U_2$$

$$v_1 A = v_2 a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = {v_2}^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$

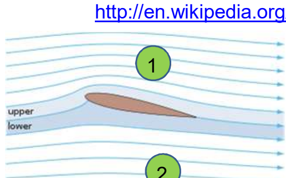
$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left({v_1}^2 - {v_2}^2\right)$$

Continuity equation connecting fluid density and velocity:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

$$\frac{\partial\rho}{\partial t}+\rho(\nabla\cdot\mathbf{v})+(\nabla\rho)\cdot\mathbf{v}=0$$

Consider: $\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+(\nabla\rho)\cdot\mathbf{v}$

$$\Rightarrow\frac{d\rho}{dt}+\rho(\nabla\cdot\mathbf{v})=0$$ alternative form of continuity equation

Some details on the velocity potential

Continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

$$\frac{\partial\rho}{\partial t}+\rho(\nabla\cdot\mathbf{v})+(\nabla\rho)\cdot\mathbf{v}=0$$

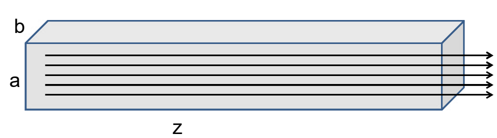
For incompressible fluid: $\rho=(\text{constant})$

$\Rightarrow\nabla\cdot\mathbf{v}=0$

Irrotational flow: $\nabla\times\mathbf{v}=0$ $\quad\Rightarrow\mathbf{v}=-\nabla\Phi$

$\Rightarrow\nabla^2\Phi=0$

Example – uniform flow



$$\nabla^2\Phi=0$$

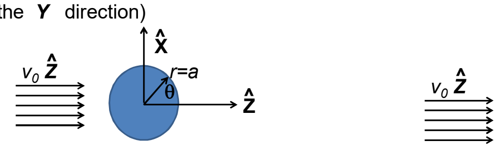
$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Possible solution:

$\Phi=-v_o z$

$\mathbf{v}=-\nabla\Phi=v_o\hat{\mathbf{z}}$

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction
Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

to be continued …