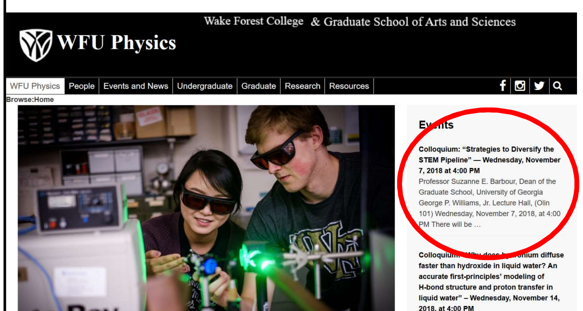
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 30: Chap. 9 of F&W

Wave equation for sound in the linear approximation

1. Sound generation
2. Sound scattering



	Fri, 10/12/2018	No class	Fall break		
19	Mon, 10/15/2018	Chap. 7	Wave equation		
20	Wed, 10/17/2018	Chap. 7	Fourier Transforms	#12	10/22/2018
21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018	Chap. 9	Mechanics of three dimensional fluids		
27	Fri, 11/02/2018	Chap. 9	Mechanics of fluids	#17	11/07/2018
28	Mon, 11/05/2018	Chap. 9	Sound waves	Project topic	
29	Wed, 11/07/2018	Chap. 9	Sound waves	#18	11/12/2018
30	Fri, 11/09/2018				
31	Mon, 11/12/2018				
32	Wed, 11/14/2018				
33	Fri, 11/16/2018				
34	Mon, 11/19/2018				
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018				

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Some comments about Monday's lecture

Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential:

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla\delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Note that:

$$\delta\mathbf{v} = -\nabla\Phi(\mathbf{r},t) = -\nabla\tilde{\Phi}(\mathbf{r},t)$$

$$\text{for } \tilde{\Phi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \int^t dt' K(t')$$

$$-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0} = K(t)$$

$$-\frac{\partial\tilde{\Phi}}{\partial t} + \frac{\delta p}{\rho_0} = 0$$

Some comments about Monday's lecture -- continued

Expressing pressure in terms of the density :

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2\delta\rho$$

$$\nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + c^2\frac{\delta\rho}{\rho_0}\right) = 0$$

$$\left(-\frac{\partial\Phi}{\partial t} + c^2\frac{\delta\rho}{\rho_0}\right) = K(t) \quad \Rightarrow -\frac{\partial^2\Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial\delta\rho}{\partial t} = 0$$

$$\frac{\partial\delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0 \quad \Rightarrow \frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Some comments about Monday's lecture -- continued

Wave equation for air:

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla\Phi$

Additional relations:

$$\delta p = c^2\delta\rho = \rho_0\frac{\partial\Phi}{\partial t}$$

$$\Rightarrow \frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2\delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}}\cdot\mathbf{V} = \hat{\mathbf{n}}\cdot\delta\mathbf{v} = -\hat{\mathbf{n}}\cdot\nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0\frac{\partial\Phi}{\partial t} = 0$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r'\int dt'\,G(\mathbf{r}-\mathbf{r}',t-t')f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}',t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

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Derivation of Green's function for wave equation -- continued

$$\text{Define: } \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t)e^{i\omega t}dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy:

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

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Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

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Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\left(R\tilde{G}(R,\omega)\right) = 0$$

$$\left(R\tilde{G}(R,\omega)\right) = A\, e^{ikR} + Be^{-ikR}$$

$$\Rightarrow \tilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

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Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

$$\text{Note that:} \quad \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

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Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

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Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}d\omega \;=\; \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t')=\frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

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➔In order to solve an inhomogenous wave equation with a time harmonic forcing term, we can use the corresponding Green's function:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

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Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right)d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot\hat{\mathbf{n}}\,d^2r$

$$\nabla^2\tilde{\Phi}+k^2\tilde{\Phi}=-\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2+k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)=-\delta(\mathbf{r}-\mathbf{r}')$$

$h\leftrightarrow\tilde{\Phi};\qquad g\leftrightarrow\tilde{G}$

$$\int_V\left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}')-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r},\omega)\right)d^3r=$$

$$\oint_S\left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r$$

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$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r'$$

If the integration volume V includes the point $\mathbf{r}=\mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\, d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r'$$

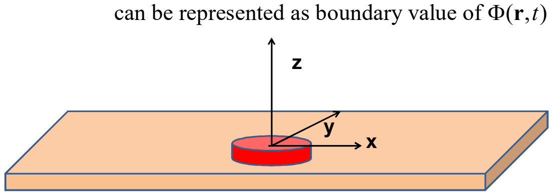
➔extra contributions from boundary

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)\, d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$

$y' = r'\sin\phi'$

For $r >> a$; $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\,e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} d\phi'\,e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'\,dr'\,J_0(kr'\sin\theta)$$

$$\int_0^w u\,du\,J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux: $\quad \mathbf{j}_e = \delta\mathbf{v}p$

Taking time average: $\quad \langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

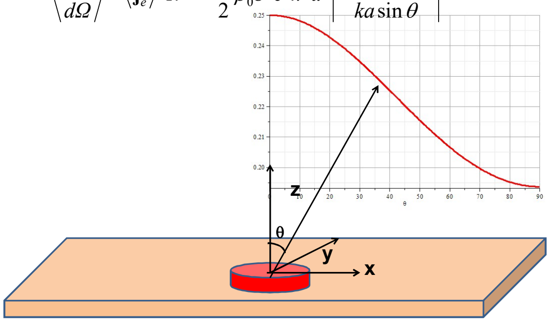
$$= \tfrac{1}{2}\rho_0 \Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$



Scattering of sound waves –
for example, from a rigid cylinder

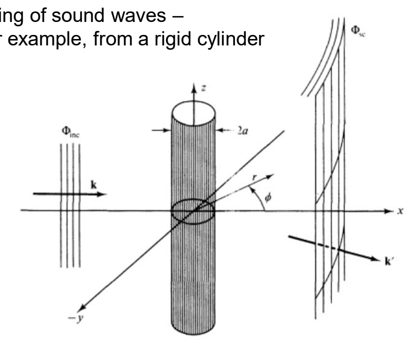


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

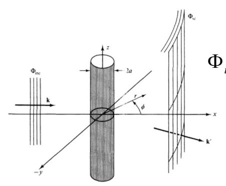


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr)$ where Hankel function

represents an outgoing wave : $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

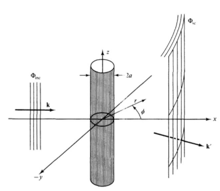


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

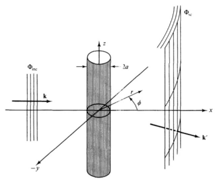
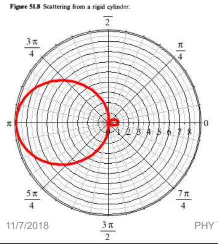


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For *ka* << *1*

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$