PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 31:

Chapter 10 in F & W: Surface waves

1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**

21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018	Chap. 9	Mechanics of three dimensional fluids		
27	Fri, 11/02/2018	Chap. 9	Mechanics of fluids	#17	11/07/2018
28	Mon, 11/05/2018	Chap. 9	Sound waves	Project topic	
29	Wed, 11/07/2018	Chap. 9	Sound waves	#18	11/12/2018
30	Fri, 11/09/2018	Chap. 9	Linear and non-linear sound		
31	Mon, 11/12/2018	Chap. 10	Surface waves	#19	11/16/2018
32	Wed, 11/14/2018	Chap. 10	Surface waves		
33	Fri, 11/16/2018	Chap. 10	Surface waves -- soliton solutions		
34	Mon, 11/19/2018	Chap. 11	Heat conductivity		
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018				
36	Wed, 11/28/2018				
37	Fri, 11/30/2018				
	Mon, 12/03/2018		Presentations I		
	Wed, 12/05/2018		Presentations II		
	Fri, 12/07/2018		Presentations III		

Colloquium: "Why does hydronium diffuse faster than hydroxide in liquid water? An accurate first-principles' modeling of H-bond structure and proton transfer in liquid water" – Wednesday, November 14, 2018, at 4:00 PM

Professor Xifan Wu,
Temple University
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, November 7, 2018, at 4:00 PM

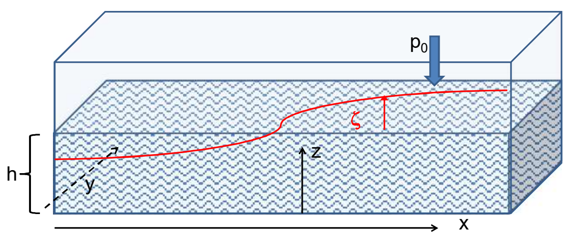
Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)

Atmospheric pressure is in equilibrium at the surface

$$p_0 = \rho g(h+\zeta)$$



Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

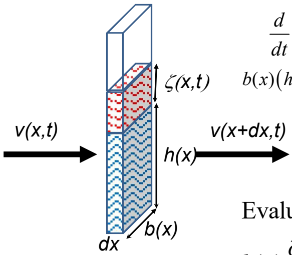
$\Rightarrow p(x,y,z,t) = p_0 + \rho g(\zeta(x,y,t) + h - z)$ within the water

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Consider a surface $\zeta(x,t)$ wave moving in the x-direction in a channel of width $b(x)$ and height $h(x)$:



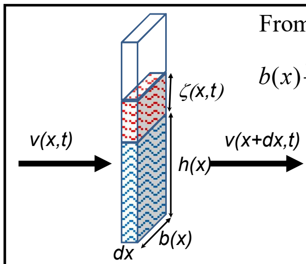
Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

$$b(x)\left(h(x)+\zeta(x,t)\right)dx \qquad b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$$

Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$



From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left((\kappa x)b_0 v(x,t)\right)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

Example continued

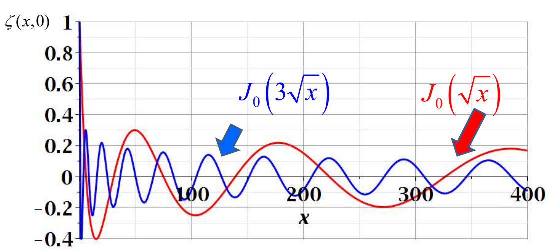
$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left(\frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x,t) = C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

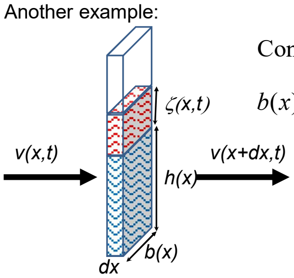
Check:

$$-\omega^2 C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

$$\zeta(x,t) = C J_0 \left(\frac{2\omega}{\sqrt{\kappa g}} \sqrt{x} \right) \cos(\omega t)$$



Another example:



Continuity condition:

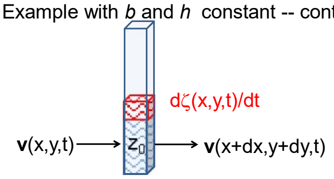
$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Special case, where *b* and *h* are constant --

For constant b and h:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$

Example with *b* and *h* constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$

For uniform channel:

Surface wave equation:

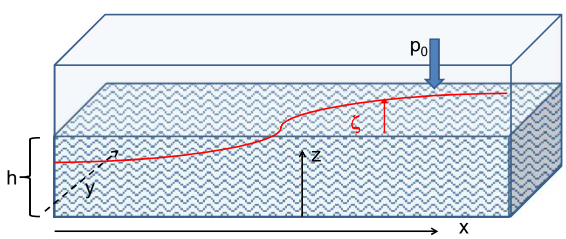
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2\nabla^2\zeta = 0 \qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)+\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=-\nabla U-\frac{\nabla p}{\rho}$$

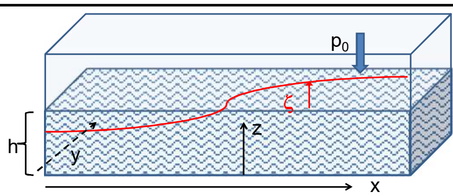
Assume that $\nabla\times\mathbf{v}=0$ (irrotational flow) $\Rightarrow \mathbf{v}=-\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}\right)=0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}=\text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v}=-\nabla^2\Phi=0$$



Within fluid: $0\le z\le h+\zeta$

$$-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+g\left(z-h\right)=\text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}\quad\text{where } v_{x,y}=v_{x,y}\left(x,y,h+\zeta,t\right)$$

Full equations:

Within fluid: $0\le z\le h+\zeta$

$$-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+g\left(z-h\right)=\text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}\quad\text{where } v_{x,y}=v_{x,y}\left(x,y,h+\zeta,t\right)$$

Linearized equations:

$$\text{For }\ 0\le z\le h+\zeta:\quad -\frac{\partial\Phi}{\partial t}+g\left(z-h\right)=0\qquad -\nabla^2\Phi=0$$

$$\text{At surface}:\ z=h+\zeta\quad \frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}=v_z\left(x,y,h+\zeta,t\right)$$

$$-\frac{\partial\Phi\left(x,y,h+\zeta,t\right)}{\partial t}+g\zeta=0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h + \zeta: \qquad \nabla^2 \Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x,z,t) = 0$$

$$\text{Consider and periodic waveform:} \quad \Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

$$\text{Boundary condition at bottom of tank:} \quad v_z(x,0,t) = 0$$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x, h+\zeta, t) = -\frac{\partial \Phi(x, h+\zeta, t)}{\partial z}$$

$$-\frac{\partial \Phi(x, h+\zeta, t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} - g\frac{\partial \Phi(x, h+\zeta, t)}{\partial z} = 0$$

$$\text{For} \quad \Phi(x, (h+\zeta), t) = A\cosh(k(h+\zeta))\cos\big(k(x-ct)\big)$$

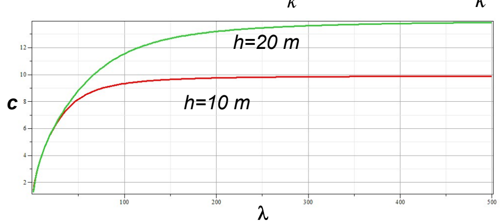
$$A\cosh(k(h+\zeta))\cos\big(k(x-ct)\big)\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} \right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta \ll h: \qquad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

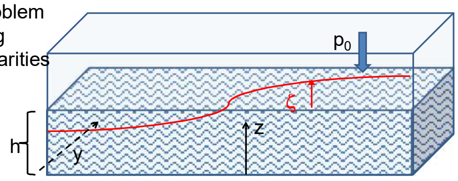
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h$, $\quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem including non-linearities



Within fluid : $\quad 0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface : $\quad z = h+\zeta \qquad$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$