## PHY 711 Classical Mechanics and Mathematical Methods
## 10-10:50 AM MWF Olin 103

**Plan for Lecture 32:**

**Chapter 10 in F & W: Surface waves**

**-- Summary of linear surface wave solutions**

**-- Non-linear contributions and soliton solutions**



This material is covered in Chapter 10 of your textbook using similar notation.



| | | | | | |
|---|---|---|---|---|---|
| 22 | Mon, 10/22/2018 | Chap. 7 | Contour integrals | | |
| 23 | Wed, 10/24/2018 | Chap. 5 | Rigid body motion | #14 | 10/26/2018 |
| 24 | Fri, 10/26/2018 | Chap. 5 | Rigid body motion | #15 | 10/31/2018 |
| 25 | Mon, 10/29/2018 | Chap. 8 | Mechanics of elastic membranes | #16 | 11/02/2018 |
| 26 | Wed, 10/31/2018 | Chap. 9 | Mechanics of three dimensional fluids | | |
| 27 | Fri, 11/02/2018 | Chap. 9 | Mechanics of fluids | #17 | 11/07/2018 |
| 28 | Mon, 11/05/2018 | Chap. 9 | Sound waves | Project topic | |
| 29 | Wed, 11/07/2018 | Chap. 9 | Sound waves | #18 | 11/12/2018 |
| 30 | Fri, 11/09/2018 | Chap. 9 | Linear and non-linear sound | | |
| 31 | Mon, 11/12/2018 | Chap. 10 | Surface waves | #19 | 11/16/2018 |
| 32 | Wed, 11/14/2018 | Chap. 10 | Surface waves -- nonlinear effects | | |
| 33 | Fri, 11/16/2018 | Chap. 11 | Heat conductivity | | |
| 34 | Mon, 11/19/2018 | Chap. 11 | Heat conductivity | | |
| | Wed, 11/21/2018 | No class | Thanksgiving holiday | | |
| | Fri, 11/23/2018 | No class | Thanksgiving holiday | | |
| 35 | Mon, 11/26/2018 | | | | |
| 36 | Wed, 11/28/2018 | | | | |
| 37 | Fri, 11/30/2018 | | | | |
| | Mon, 12/03/2018 | | Presentations I | | |
| | Wed, 12/05/2018 | | Presentations II | | |
| | Fri, 12/07/2018 | | Presentations III | | |

**Colloquium: "Why does hydronium diffuse faster than hydroxide in liquid water? An accurate first-principles' modeling of H-bond structure and proton transfer in liquid water" – Wednesday, November 14, 2018, at 4:00 PM**

Posted on November 9, 2018

Professor Xifan Wu,
Temple University
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, November 7, 2018, at 4:00 PM

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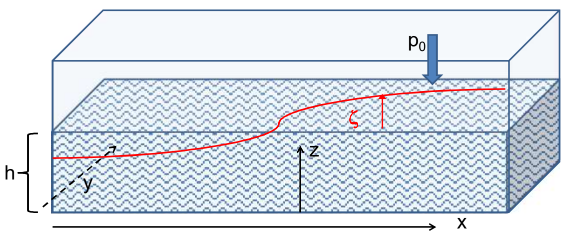
Consider a container of water with average height h and surface h+ζ(x,y,t)

Atmospheric pressure $p_0$ is in equilibrium at the surface

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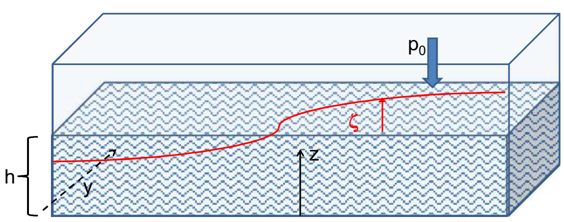


Euler's equation for incompressible fluid

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla\left(-\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho}\right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

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Keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h + \zeta$ : $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t) = 0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\left(k(x-ct)\right)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2\right)Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

At surface: $z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z}$

Also: $-\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z} = 0$$



Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$

$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

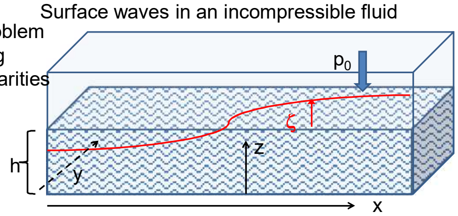
$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additional, you text considers the effects of surface tension.



Surface waves in an incompressible fluid

General problem including non-linearities



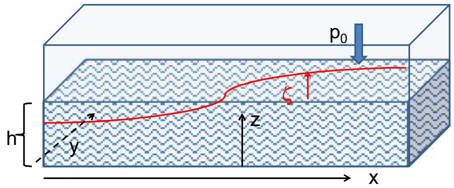

Within fluid: $0 \le z \le h + \zeta$

$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}v^2 + g\left(z-h\right) = \text{constant} \qquad \Phi = \Phi(x,y,z,t)$

$-\nabla^2\Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x,y,z,t) = -\nabla\Phi(x,y,z,t)$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x\frac{\partial \zeta}{\partial x} + v_y\frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$$




Further simplifications; assume trivial $y$ - dependence

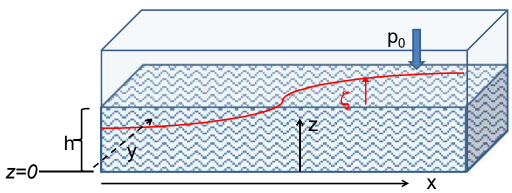
$\Phi = \Phi(x,z,t)$ $\qquad \zeta = \zeta(x,t)$

Within fluid : $\quad 0 \le z \le h + \zeta$

At surface : $\quad v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$



Non-linear effects in surface waves:




Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$



**Detailed analysis of non-linear surface waves**

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \le z \le h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$



Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$



**Analysis assuming water height *z* is small relative to variations in the direction of wave motion** *(x)*
**Taylor's expansion about** *z* = 0:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom ensures that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the

Taylor expansion . In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Check linearized equations and their solutions:
Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,h,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t}$$

Decoupled equations: $\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}.$

➔linear wave equation with $c^2=gh$



Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2 + \left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2\right]_{z=h+\zeta} + g\zeta(x,t) = 0.$$

Consistency of surface velocity

$$-\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta} = 0$$



Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

The expressions keep the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms.

Coupled equations: $-\frac{\partial \phi}{\partial t}+\frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2+g\zeta=0.$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right)-\frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4}-\frac{\partial \zeta}{\partial t}=0.$$

Traveling wave solutions with new notation:

$u \equiv x-ct \qquad \phi(x,t)\equiv \chi(u) \qquad$ and $\quad \zeta(x,t)\equiv \eta(u)$

Note that the wave "speed" *c* will be consistently determined

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$



Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\Rightarrow \chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow (h+\eta)\chi'-\frac{h^3}{6}\chi'''+c\eta=0$$



Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of *η(u):*

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3 g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Note: $\quad c^2=gh+\ldots$

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of $\eta$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

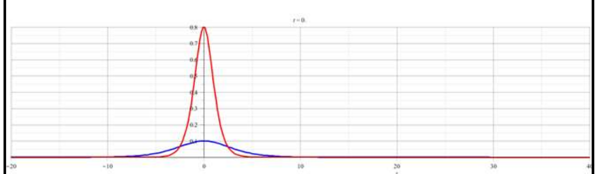
Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad \text{where } \eta_0 \text{ is a constant}$$



$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$





Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta=2\eta_0,\quad \bar{x}=\sqrt{\frac{3}{2h}}\frac{x}{h},\quad \text{and}\quad \bar{t}=\sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial\eta}{\partial\bar{t}}+6\eta\frac{\partial\eta}{\partial\bar{x}}+\frac{\partial^3\eta}{\partial\bar{x}^3}=0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t})=\frac{\beta}{2}\text{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x}-\beta\bar{t})\right].$$

## More details

Modified surface amplitude equation in terms of $\eta$ :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2};\quad \frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du};\quad \frac{\partial\eta}{\partial x}=\frac{d\eta}{du}.$

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of $x$ and $t$:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$ :

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$



Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}\quad\Rightarrow\frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)\quad\Rightarrow\frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt $u$ and assume solution vanishes for $u\to\infty$

$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du\qquad\Rightarrow\eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$



## Summary

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad\text{where } \eta_0 \text{ is a constant}$$

Photo of canal soliton http://www.ma.hw.ac.uk/solitons/