# PHY 711 Classical Mechanics and Mathematical Methods
## 10-10:50 AM MWF Olin 103

## Plan for Lecture 34

**Physics of elastic continua – Chap. 13 in F & W**

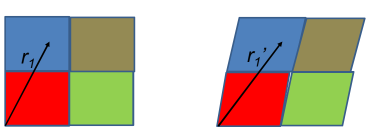
1. Stress and strain
2. Waves in elastic media



| | | | | | |
|---|---|---|---|---|---|
| 23 | Wed, 10/24/2018 | Chap. 5 | Rigid body motion | #14 | 10/26/2018 |
| 24 | Fri, 10/26/2018 | Chap. 5 | Rigid body motion | #15 | 10/31/2018 |
| 25 | Mon, 10/29/2018 | Chap. 8 | Mechanics of elastic membranes | #16 | 11/02/2018 |
| 26 | Wed, 10/31/2018 | Chap. 9 | Mechanics of three dimensional fluids | | |
| 27 | Fri, 11/02/2018 | Chap. 9 | Mechanics of fluids | #17 | 11/07/2018 |
| 28 | Mon, 11/05/2018 | Chap. 9 | Sound waves | Project topic | |
| 29 | Wed, 11/07/2018 | Chap. 9 | Sound waves | #18 | 11/12/2018 |
| 30 | Fri, 11/09/2018 | Chap. 9 | Linear and non-linear sound | | |
| 31 | Mon, 11/12/2018 | Chap. 10 | Surface waves | #19 | 11/16/2018 |
| 32 | Wed, 11/14/2018 | Chap. 10 | Surface waves -- nonlinear effects | | |
| 33 | Fri, 11/16/2018 | Chap. 11 | Heat conductivity | #20 | 11/26/2018 |
| 34 | Mon, 11/19/2018 | Chap. 13 | Elastic media | | |
| | Wed, 11/21/2018 | No class | Thanksgiving holiday | | |
| | Fri, 11/23/2018 | No class | Thanksgiving holiday | | |
| 35 | Mon, 11/26/2018 | Chap. 12 | Viscous fluids | | |
| 36 | Wed, 11/28/2018 | Chap. 12 | Viscous fluids | | |
| 37 | Fri, 11/30/2018 | | Review | | |
| | Mon, 12/03/2018 | | Presentations I | | |
| | Wed, 12/05/2018 | | Presentations II | | |
| | Fri, 12/07/2018 | | Presentations III | | |



Brief introduction to elastic continua




reference deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1) + \ldots$$

Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \quad \epsilon_{ij} \quad + \quad O_{ij}$$

elastic strain tensor

rotation of material



Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

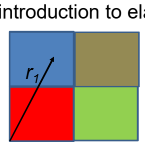
$$\equiv \epsilon_{ij} \quad + \quad O_{ij}$$

$V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $\quad V' = \mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')$ $\quad V' = V(1 + \nabla \cdot \mathbf{u}) = V(1 + \mathrm{Tr}(\epsilon))$

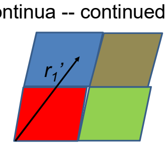
$$\nabla \cdot \mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{d\rho}{\rho}$$



Brief introduction to elastic continua -- continued




reference




deformation

$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1)$ $\quad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla\right)\mathbf{u}(\mathbf{r}_1) + \ldots$$

$$x_{2i}' - x_{1i}' \approx x_{2i} - x_{1i} + \sum_{j=1}^{3} \epsilon_{ij}\left(x_{2j} - x_{1j}\right)$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Effects of strain on a vector:

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$a' = |\mathbf{a}' \cdot \mathbf{a}'| \approx a\left(1 + \epsilon_{11}\right)$$

## Deformation


$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$\mathbf{b}' = \mathbf{b} + b\left(\epsilon_{12}\hat{\mathbf{x}} + \epsilon_{22}\hat{\mathbf{y}} + \epsilon_{32}\hat{\mathbf{z}}\right)$$

$$\text{for } \mathbf{a}\cdot\mathbf{b} = 0 = ab\cos\theta \qquad \Rightarrow \theta = \frac{\pi}{2}$$

$$\mathbf{a}'\cdot\mathbf{b}' \approx ab\left(\epsilon_{21} + \epsilon_{12}\right) = 2ab\epsilon_{12} = ab\cos\theta'$$

$$\cos\theta' = \cos\left(\theta + \left(\theta' - \theta\right)\right) = \cos\theta\,\cos\left(\theta'-\theta\right) - \sin\theta\,\sin\left(\theta'-\theta\right)$$

$$\approx -\sin\theta\,\sin\left(\theta'-\theta\right) \approx -\left(\theta'-\theta\right)$$

$$\theta' \approx \theta - 2\epsilon_{12} = \frac{\pi}{2} - 2\epsilon_{12}$$



## Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij}\,dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}}dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $F_x = -kx$:

$$\text{Lame' coefficients}: \quad T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\text{or}: \quad T_{ij} = -\lambda\delta_{ij}\,\mathrm{Tr}\left(\epsilon\right) - 2\mu\epsilon_{ij}$$



## Elastic stress tensor -- continued

$$T_{ij} = -\lambda\delta_{ij}\,\mathrm{Tr}\left(\epsilon\right) - 2\mu\epsilon_{ij}$$

Note that: $\mathrm{Tr}\left(T\right) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}\,\mathrm{Tr}\left(\epsilon\right)$

$$K \equiv \text{bulk modulus} = -V\left(\frac{\partial p}{\partial V}\right)$$

Inverse Hooke's law: $\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr\left(T\right)\right)$

Stress tensor -- continued

$$\epsilon_{ij} = -\frac{1}{2\mu}\left( T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\, \delta_{ij} Tr(T) \right)$$

In terms of bulk modulus: $K = \lambda + \frac{2}{3}\mu$

$$\lambda = K - \frac{2}{3}\mu$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left( T_{ij} - \frac{1}{3}\, \delta_{ij} Tr(T) \right)$$



$$\epsilon_{ij} = -\frac{1}{2\mu}\left( T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\, \delta_{ij} Tr(T) \right) = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left( T_{ij} - \frac{1}{3}\, \delta_{ij} Tr(T) \right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij} dp$

$$\mathrm{Tr}(T) = 3dp$$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv -\frac{dp}{3K}\delta_{ij}$$

Note that: $\mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{dp}{K}$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$



$$\epsilon_{ij} = -\frac{1}{2\mu}\left( T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\, \delta_{ij} Tr(T) \right) = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left( T_{ij} - \frac{1}{3}\, \delta_{ij} Tr(T) \right)$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$\epsilon_{zz} = -\frac{1}{E}T_{zz}$ in terms of Young's modulus

$$E = \frac{9K\mu}{3K + \mu}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\left(\frac{1}{9K} - \frac{1}{6\mu}\right)dp$$

Poisson ratio: $\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{\epsilon_{yy}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

Shear modulus

$$T_{ij} = \begin{cases} -f & \text{for } T_{xy} \text{ or } T_{yx} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{f}{2\mu}$$



Relationships between the elastic moduli:

Poisson's ratio:

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$

Young's modulus:

$$E = \frac{9K\mu}{3K + \mu}$$

Shear modulus: $\mu$

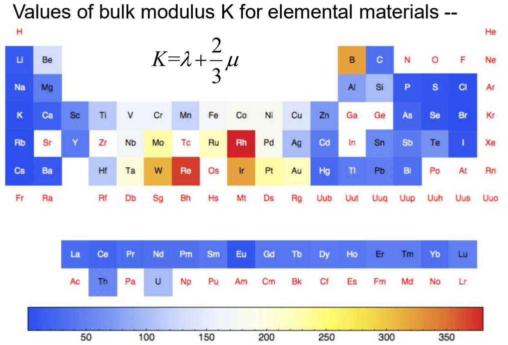
Relationships between elastic constants:

$$K = \frac{1}{3}\frac{E}{1 - 2\sigma}$$

$$\mu = \frac{1}{2}\frac{E}{1 + \sigma}$$



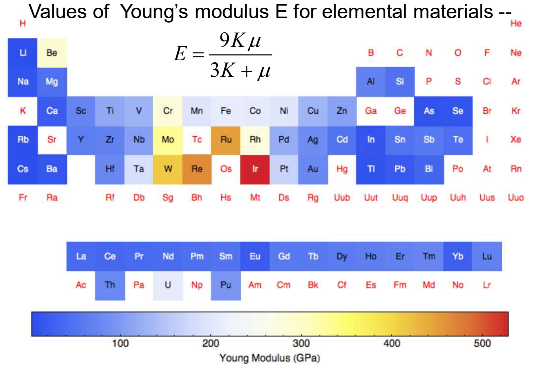
Values of bulk modulus K for elemental materials --




Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

## Values of Young's modulus E for elemental materials --

$$E = \frac{9K\mu}{3K+\mu}$$

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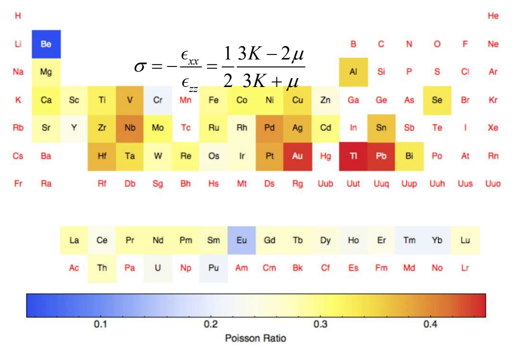
## Values of Poisson ratio σ for elemental materials --

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$$

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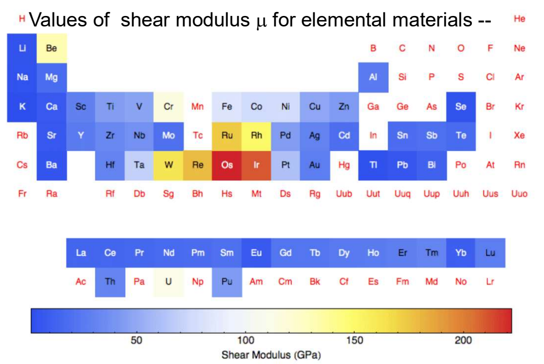
## Values of shear modulus μ for elemental materials --




Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Summary -- Elastic stress tensor

$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th}$ component of force acting on surface $\hat{\mathbf{n}}dA \equiv d\mathbf{A}$

Generalization of Hooke's law, $F_x = -kx$:

Lame' coefficients : $T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$

material-dependent empirical parameters

Bulk modulus: $K = \lambda + \frac{2}{3}\mu$

Young's modulus: $E = \frac{9K\mu}{3K+\mu}$

Poisson ratio: $\sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$

Shear modulus: $\mu$



Stress, Strain

Hooke's Law: $T_{ij} = -K\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}(\epsilon)\right)$

Inverse Hooke's Law: $\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\delta_{ij}Tr(T)\right)$

Elastic force: $F_i^{\text{elastic}} = \sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}$

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\, \mathbf{F}^{\text{elastic}}\cdot\delta\mathbf{u}$$

$$= \int_V d^3x \sum_{ij=1}^{3}\frac{\partial T_{ij}}{\partial x_j}\delta u_i$$

$$= \sum_{ij=1}^{3}\int_A dA_j\, T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\frac{\partial \delta u_i}{\partial x_j}$$



Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\, \mathbf{F}^{\text{elastic}}\cdot\delta\mathbf{u}$$

$$= \sum_{ij=1}^{3}\int_A dA_j\, T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\frac{\partial \delta u_i}{\partial x_j}$$

$$= \sum_{ij=1}^{3}\int_A dA_j\, T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\epsilon_{ij}$$

surface contribution — bulk contribution

For large samples: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3}\int_V d^3x T_{ij}\epsilon_{ij}$

Bulk elastic energy: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3} \int_V d^3x T_{ij} \epsilon_{ij}$

Integrating from 0 to final strain $\epsilon_{ij}$ :

$$\delta W^{\text{elastic}} = -\frac{1}{2} \sum_{ij=1}^{3} T_{ij} \epsilon_{ij}$$

Hooke's Law: $T_{ij} = -K\delta_{ij} \text{Tr}(\epsilon) - 2\mu \left( \epsilon_{ij} - \frac{1}{3} \delta_{ij} \text{Tr}(\epsilon) \right)$

$$\delta W^{\text{elastic}} = \frac{1}{2} K (\text{Tr } \epsilon)^2 + \mu \sum_{ij=1}^{3} \left( \epsilon_{ij} - \frac{\delta_{ij}}{3} \text{Tr } \epsilon \right)^2$$

$$= \frac{1}{2} \lambda (\text{Tr } \epsilon)^2 + \mu \sum_{ij=1}^{3} \epsilon_{ij}^2$$



Note that the two relations:

$$\delta W^{\text{elastic}} = \frac{1}{2} K (\text{Tr } \epsilon)^2 + \mu \sum_{ij=1}^{3} \left( \epsilon_{ij} - \frac{\delta_{ij}}{3} \text{Tr } \epsilon \right)^2$$

$$T_{ij} = -K\delta_{ij} \text{Tr}(\epsilon) - 2\mu \left( \epsilon_{ij} - \frac{1}{3} \delta_{ij} \text{Tr}(\epsilon) \right)$$

Ensure that: $\frac{\partial \delta W^{\text{elastic}}}{\partial \epsilon_{ij}} = -T_{ij}$



Dynamical equations of motion

Recall Newton's second law for continuum system with density $\rho$, velocity components $v_k$ and stress tensor $T_{ij}$ :

$$\frac{\partial (\rho v_k)}{\partial t} = -\sum_{l=1}^{3} \frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

external force density

For our elastic medium: $\rho$ does not vary in time

Velocity related to displacement: $v_k = \frac{\partial u_k}{\partial t}$

Hooke's law: $T_{kl} = -K\delta_{kl} \text{Tr}(\epsilon) - 2\mu \left( \epsilon_{kl} - \frac{1}{3} \delta_{kl} \text{Tr}(\epsilon) \right)$

$$= -K\delta_{kl} (\nabla \cdot \mathbf{u}) - 2\mu \left( \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) - \frac{1}{3} \delta_{kl} (\nabla \cdot \mathbf{u}) \right)$$

For: $T_{kl} = -K\delta_{kl}(\nabla\cdot\mathbf{u}) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right) - \frac{1}{3}\delta_{kl}(\nabla\cdot\mathbf{u})\right)$

$$\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} = -\left(K-\frac{2}{3}\mu\right)\sum_{l=1}^{3}\left(\delta_{kl}\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_l}\right) - \mu\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$= -\left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$\frac{\partial(\rho v_k)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

$$\rho\frac{\partial^2 u_k}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} + \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l} + \rho f_k$$

Vector form: $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}$



Dynamical equations of elastic continuum

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \rho\mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla\times\mathbf{u}_l = 0$ and $\nabla\cdot\mathbf{u}_t = 0$

$$c_l = \left(\frac{K+\frac{4}{3}\mu}{\rho}\right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho}\right)^{1/2}$$



Typical velocities of longitudinal sound waves

http://www.engineeringtoolbox.com/sound-speed-solids-d_713.html

air: $c_l$=343 m/s
water: $c_l$=1433 m/s

| Material | $c_l$ (m/s) |
|---|---|
| Aluminum, shear - longitudinal wave | 3100 - 6400 |
| Beryllium | 12890 |
| Brass | 3475 |
| Brick | 4176 |
| Concrete | 3200 - 3600 |
| Copper | 4600 |
| Cork | 366 - 518 |
| Diamond | 12000 |
| Glass | 3962 |
| Glass, Pyrex | 5640 |
| Gold | 3240 |
| Granite | 5950 |
| Hardwood | 3962 |
| Iron | 5130 |
| Lead | 1960 - 2160 |
| Lucite | 2680 |
| Rubber, butyl | 1830 |
| Rubber | 40 - 150 |
| Silver | 3650 |
| Steel | 6100 |
| Steel, stainless | 5790 |
| Titanium | 6070 |
| Wood (hard) | 3960 |
| Wood | 3300 - 3600 |

from:
https://pangea.stanford.edu/courses/gp262/Notes/5.Elasticity.pdf

| Mineral | Density | Young's Modulus | Bulk Modulus | Shear Modulus | Vp | Vs | Poisson's Ratio |
|---|---|---|---|---|---|---|---|
| Quartz | 2.6500 | 95.756 | 36.600 | 45.000 | 6.0376 | 4.1208 | 0.063953 |
| Calcite | 2.7100 | 84.293 | 76.800 | 32.000 | 6.6395 | 3.4363 | 0.31707 |
| Dolomite | 2.8700 | 116.57 | 94.900 | 45.000 | 7.3465 | 3.9597 | 0.29527 |
| Clay (kaolinite) | 1.5800 | 3.2034 | 1.5000 | 1.4000 | 1.4597 | 0.94132 | 0.14407 |
| Muscovite | 2.7900 | 100.84 | 61.500 | 41.100 | 6.4563 | 3.8381 | 0.22673 |
| Feldspar (Albite) | 2.6300 | 69.010 | 75.600 | 25.600 | 6.4594 | 3.1199 | 0.34786 |
| Halite | 2.1600 | 37.242 | 24.800 | 14.900 | 4.5474 | 2.6264 | 0.24972 |
| Anhydrite | 2.9800 | 74.431 | 56.100 | 29.100 | 5.6432 | 3.1249 | 0.27888 |
| Pyrite | 4.9300 | 305.85 | 147.40 | 132.50 | 8.1076 | 5.1842 | 0.15417 |
| Siderite | 3.9600 | 134.51 | 123.70 | 51.000 | 6.9576 | 3.5887 | 0.31876 |
| gas | 0.00065000 | 0.0000 | 0.00013000 | 0.0000 | 0.44721 | 0.0000 | 0.50000 |
| water | 1.0000 | 0.0000 | 2.2500 | 0.0000 | 1.5000 | 0.0000 | 0.50000 |
| oil | 0.80000 | 0.0000 | 1.0200 | 0.0000 | 1.1292 | 0.0000 | 0.50000 |

C.1

**densities in g/cm³**
**moduli in GPa**
**velocities in km/s**



Dynamical equations of elastic continuum

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left( K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u}) + \rho \mathbf{f}$$

In absense of external force:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left( K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u})$$

Suppose: $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla \times \mathbf{u}_l = 0$ and $\nabla \cdot \mathbf{u}_t = 0$

$$\rho \frac{\partial^2 (\mathbf{u}_l + \mathbf{u}_t)}{\partial t^2} = \mu \nabla^2 (\mathbf{u}_l + \mathbf{u}_t) + \left( K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u}_l)$$



Dynamical equations of elastic continuum
Transverse component:

$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $c_t = \sqrt{\frac{\mu}{\rho}}$

Longitudinal component: $\rho \frac{\partial^2 \mathbf{u}_l}{\partial t^2} = \mu \nabla^2 \mathbf{u}_l + \left( K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u}_l)$

Note that the longitudinal wave has its displacement along its propagation direction $x_p$, so that $\mathbf{u}_l = \mathbf{u}_l(x_p) \equiv u_l(x_p)\hat{\mathbf{x}}_p$

$$\Rightarrow \rho \frac{\partial^2 u_l}{\partial t^2} = \mu \frac{\partial^2 u_l}{\partial x_p^2} + \left( K + \frac{1}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2} = \left( K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

Dynamical equations of elastic continuum
Longitudinal component -- continued:

$$\rho \frac{\partial^2 u_l}{\partial t^2} = \left(K + \frac{4}{3}\mu\right)\frac{\partial^2 u_l}{\partial x_p^2}$$

$$\frac{\partial^2 u_l}{\partial t^2} = \left(\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho}\right)\frac{\partial^2 u_l}{\partial x_p^2} \equiv c_l^2 \frac{\partial^2 u_l}{\partial x_p^2}$$

Longitudinal wave velocity: $c_l = \sqrt{\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho}}$

Transverse component:

$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho}\nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $c_t = \sqrt{\frac{\mu}{\rho}}$



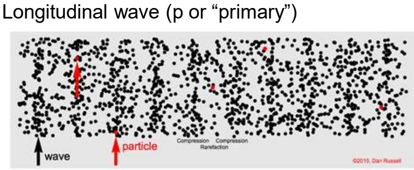
Some values:

| Substance | Density ($g/cm^3$) | $V_l$ (m/s) | $V_s$ (m/s) |
|---|---|---|---|
| Metals | | | |
| Aluminum, rolled | 2.7 | 6420 | 3040 |
| Berylium | 1.87 | 12890 | 8880 |
| Brass (70 Cu, 30 Zn) | 8.6 | 4700 | 2110 |
| Copper, annealed | 8.93 | 4760 | 2325 |
| Copper, rolled | 8.93 | 5010 | 2270 |
| Gold, hard-drawn | 19.7 | 3240 | 1200 |
| Iron, Armco | 7.85 | 5960 | 3240 |
| Lead, annealed | 11.4 | 2160 | 700 |
| Lead, rolled | 11.4 | 1960 | 690 |
| Molybdenum | 10.1 | 6250 | 3350 |
| Monel metal | 8.90 | 5350 | 2720 |
| Nickel (unmagnetized) | 8.85 | 5480 | 2990 |
| Nickel | 8.9 | 6040 | 3000 |
| Platinum | 21.4 | 3260 | 1730 |
| Silver | 10.4 | 3650 | 1610 |
| Steel, mild | 7.85 | 5960 | 3235 |
| Steel, 347 Stainless | 7.9 | 5790 | 3100 |

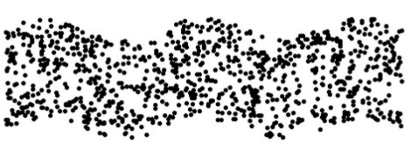
| Substance | Density ($g/cm^3$) | $V_l$ (m/s) | $V_s$ (m/s) |
|---|---|---|---|
| Tin, rolled | 7.3 | 3320 | 1670 |
| Titaninum | 4.5 | 6070 | 3125 |
| Tungsten, annealed | 19.3 | 5220 | 2890 |
| Tungsten Carbide | 13.8 | 6655 | 3980 |
| Zinc, rolled | 7.1 | 4210 | 2440 |
| Various | | | |
| Fused silica | 2.2 | 5968 | 3764 |
| Glass, pyrex | 2.32 | 5640 | 3280 |
| Glass, heavy silicate flint | 3.88 | 3980 | 2380 |
| Lucite | 1.18 | 2680 | 1100 |
| Nylon 6-6 | 1.11 | 2620 | 1070 |
| Polyethylene | 0.90 | 1950 | 540 |
| Polystyrene | 1.06 | 2350 | 1120 |



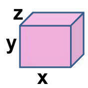
Animations from website:
https://www.acs.psu.edu/drussell/demos/waves/wavemotion.html
Longitudinal wave (p or "primary")




Transverse wave (s or "secondary")



Elasticity in solids




We imagine that near equilibrium, the solid can be described in terms of a potential function $\phi(\{\mathbf{R}\})$ where $\{\mathbf{R}\}$ represents the positions of each atom:

$$E=\sum_{\mathbf{R}}\phi(\{\mathbf{R}\})+\frac{1}{2}\sum_{\mathbf{RR}'}(\mathbf{u}(\mathbf{R})-\mathbf{u}(\mathbf{R}'))\cdot\nabla\phi(\{\mathbf{R}\})+\frac{1}{4}\sum_{\mathbf{RR}'}\left((\mathbf{u}(\mathbf{R})-\mathbf{u}(\mathbf{R}'))\cdot\nabla\right)^2\phi(\{\mathbf{R}\})+\ldots$$

vanishes at equilibrium

$$\delta W^{\text{elastic}}=\frac{1}{4}\sum_{\substack{\mathbf{RR}'\\ ij}}\left(u_i(\mathbf{R})-u_i(\mathbf{R}')\right)\frac{\partial^2\phi(\{\mathbf{R}\})}{\partial u_i\partial u_j}\left(u_j(\mathbf{R})-u_j(\mathbf{R}')\right)$$



$$\delta W^{\text{elastic}}=\frac{1}{4}\sum_{\substack{\mathbf{RR}'\\ ij}}\left(u_i(\mathbf{R})-u_i(\mathbf{R}')\right)\frac{\partial^2\phi(\{\mathbf{R}\})}{\partial u_i\partial u_j}\left(u_j(\mathbf{R})-u_j(\mathbf{R}')\right)$$

Note that $\mathbf{u}(\mathbf{R}')\approx\mathbf{u}(\mathbf{R}')+(\mathbf{R}'-\mathbf{R})\cdot\nabla\mathbf{u}(\mathbf{R})$

In terms of strain coeffients $\epsilon_{ij}$ :

$$\delta W^{\text{elastic}}=\frac{1}{2}\sum_{ijkl}\epsilon_{ij}c_{ijkl}\epsilon_{kl}$$

where coeffients $c_{ijkl}$ are composed of permutations of $R_i\frac{\partial^2\phi}{\partial u_j\partial u_k}R_l$

For the most general case $c_{ijkl}$ have 21 distrinct terms, for a cube there are only 3 unique terms.



Simplified notation:

$xx\to 1$
$yy\to 2$
$zz\to 3$
$yz\to 4$
$zx\to 5$
$xy\to 6$

For cubic crystals, the unique coefficients are:

$C_{11}=c_{xxxx}$ $\quad C_{12}=c_{xxyy}$ $\quad C_{44}=c_{yzyz}$

Some typical values (Ref. Ashcroft and Mermin (1976))

| | $C_{11}$ (GPa) | $C_{12}$(GPa) | $C_{44}$(GPa) |
|---|---|---|---|
| Na | 7.0 | 6.1 | 4.5 |
| Al | 107 | 61 | 28 |
| Fe | 234 | 136 | 118 |
| Si | 166 | 64 | 80 |
| NaCl | 48.7 | 12.4 | 12.6 |