PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 36

Viscous fluids – Chap. 12 in F & W

1. Navier-Stokes equation
2. Derivation of Stoke's law
3. Effects of viscosity on sound waves

21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018	Chap. 9	Mechanics of three dimensional fluids		
27	Fri, 11/02/2018	Chap. 9	Mechanics of fluids	#17	11/07/2018
28	Mon, 11/05/2018	Chap. 9	Sound waves	Project topic	
29	Wed, 11/07/2018	Chap. 9	Sound waves	#18	11/12/2018
30	Fri, 11/09/2018	Chap. 9	Linear and non-linear sound		
31	Mon, 11/12/2018	Chap. 10	Surface waves	#19	11/16/2018
32	Wed, 11/14/2018	Chap. 10	Surface waves -- nonlinear effects		
33	Fri, 11/16/2018	Chap. 11	Heat conductivity	#20	11/26/2018
34	Mon, 11/19/2018	Chap. 13	Elastic media		
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018	Chap. 12	Viscous fluids		
36	Wed, 11/28/2018	Chap. 12	Viscous fluids		
37	Fri, 11/30/2018		Review		
	Mon, 12/03/2018		Presentations I		
	Wed, 12/05/2018		Presentations II		
	Fri, 12/07/2018		Presentations III		

Comments on presentation schedule:

PHY 711 Presentation Schedule for Fall 2018

Monday December 3, 2018

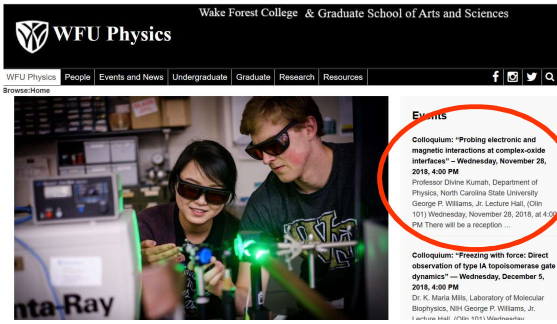
	Name	Topic
10-10:25	Leda Gao	Foucoult Pendulum
10:25-10:50		

Wednesday December 5, 2018

	Name	Topic
10-10:15	Dizhou Wu	Three Body
10:15-10:30	Eric Grotzke	TBD
10:30-10:45	Daniel Vickers	Inelastic Collisions in Interstellar Medium

Friday December 7, 2018

	Name	Topic
10-10:15	Lindsey Gray	Molecular vibrations
10:15-10:30	Ryan Sullivan	Chaotic Pendula
10:30-10:45	Shohreh	Vibrational modes in two dimensions



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Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Typical viscosities at 20° C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

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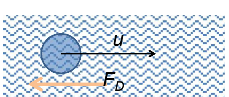
Viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$



Plan:

1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

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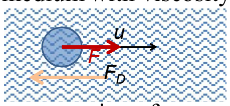
Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η :

$$F_D = -\eta(6\pi R u)$$



Effects of drag force on motion of particle of mass m with constant force F :

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$

Effects of drag force on motion of particle of mass m with constant force F :

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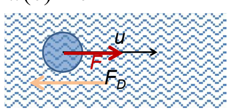
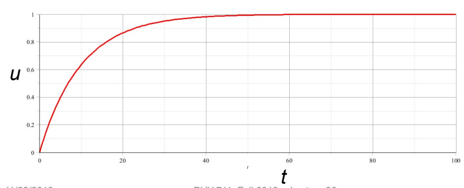
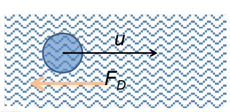


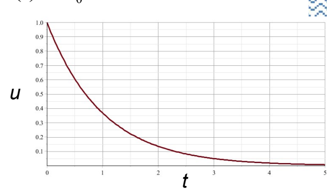


Effects of drag force on motion of particle of mass m with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$



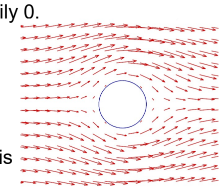


Recall: PHY 711 -- Assignment #17 Nov. 2, 2018

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all *r* > *a*. Assume that for *r* = *a,* the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$



Note: in this case, no net pressure is exerted on the sphere.

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Continuity equation: $\nabla\cdot\mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0$;

$$\Rightarrow \nabla p = \eta\nabla^2\mathbf{v}$$

$$\nabla p = \eta\nabla^2\mathbf{v}$$

Take curl of both sides of equation:

$$\nabla\times(\nabla p) = 0 = \eta\nabla^2(\nabla\times\mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla\times(\nabla\times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r)\xrightarrow[r\to\infty]{} 0$

Note that:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u} \right) + \mathbf{u}$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) = \nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = \nabla \times \left(\nabla \times \left(\nabla \times \mathbf{A} \right) \right) = -\nabla \times \left(\nabla^2 \mathbf{A} \right)$$

Also note: $\nabla p = \eta \nabla^2 \mathbf{v}$

$$\Rightarrow \nabla \times \nabla p = \nabla \times \eta \nabla^2 \mathbf{v} \qquad \text{or} \qquad \nabla^2 \left(\nabla \times \mathbf{v} \right) = 0$$

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$$\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u} \right) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}} \right) = \nabla \left(\nabla \cdot f(r)\hat{\mathbf{z}} \right) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \nabla^2 \left(\nabla \times \mathbf{v} \right) = 0$$

$$\nabla^4 \left(\nabla \times f(r)\hat{\mathbf{z}} \right) = 0 \qquad \Rightarrow \nabla^4 \left(\nabla f(r) \times \hat{\mathbf{z}} \right) = 0 \qquad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta \left(1 - \frac{2}{r}\frac{df}{dr} \right) = u\cos\theta \left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3} \right)$$

$$v_\theta = -u\sin\theta \left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr} \right) = -u\sin\theta \left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3} \right)$$

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Some details:

$$\nabla^4 f(r) = 0 \qquad \Rightarrow \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right)^2 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$\mathbf{v} = u \left(\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}} \right) + \hat{\mathbf{z}} \right)$$

$$= u \left(\nabla \left(\nabla \cdot \left(f(r)\hat{\mathbf{z}} \right) \right) - \nabla^2 f(r)\hat{\mathbf{z}} + \hat{\mathbf{z}} \right)$$

Note that: $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{v} = u \left(\nabla \left(\frac{df}{dr}\cos\theta \right) - \left(\nabla^2 \left(f(r) \right) - 1 \right) \left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \right) \right)$$

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$$v_r = u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1-4C_1-\frac{2C_2}{r}+\frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{d^2f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1-4C_1-\frac{C_2}{r}-\frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r\to\infty)=\mathbf{u}$: $\quad \Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R)=0$ solve for C_2, C_4; $\quad C_2=\frac{3R}{4},\ C_4=\frac{R^3}{4}$

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

$$f(r) = C_3 + \frac{3R}{4}r + \frac{R^3}{4r} \qquad \mathbf{v} = u\left(\nabla\left(\nabla\cdot\left(f(r)\hat{\mathbf{z}}\right)\right) - \nabla^2 f(r)\hat{\mathbf{z}} + \hat{\mathbf{z}}\right)$$

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right) \qquad v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = \eta u\left(\nabla\left(\nabla\cdot\left(\nabla^2 f(r)\hat{\mathbf{z}}\right)\right) - \nabla^4 f(r)\hat{\mathbf{z}}\right) = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

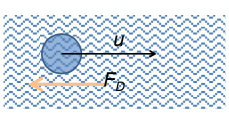
$$\Rightarrow p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

$$p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

Corresponds to:

$$F_D\cos\theta = \left(p(R)-p_0\right)4\pi R^2$$

$$\Rightarrow \mathrm{F}_D = -\eta u\left(6\pi R\right)$$



Recap --

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv p_0 + c^2\delta\rho$$

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Sound waves without viscosity -- continued

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \Rightarrow \omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0 \qquad \Rightarrow -\omega\delta\rho_0 + \rho_0\mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}}\cdot\delta\mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound
Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Assume: $\mathbf{v}=0+\delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho=\rho_0+\delta\rho$

$$p(s,\rho)=p_0+\delta p=p_0+c^2\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\delta s$$

where $c^2\equiv\left(\frac{\partial p}{\partial \rho}\right)_s$

viscosity causes heat transfer

Newton-Euler equations for viscous fluids – effects on sound
Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad \rho T\frac{\partial s}{\partial t}=k_{th}\nabla^2 T$$

Assume: $\mathbf{v}=0+\delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho=\rho_0+\delta\rho$

$$p=p_0+\delta p=p_0+c^2\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\delta s \quad \text{where } c^2\equiv\left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T=T_0+\delta T=T_0+\left(\frac{\partial T}{\partial \rho}\right)_s\delta\rho+\left(\frac{\partial T}{\partial s}\right)_\rho\delta s$$

$$s=s_0+\delta s$$

Newton-Euler equations for viscous fluids –
linearized equations

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\Rightarrow\frac{\partial \delta\mathbf{v}}{\partial t}=-\underbrace{\frac{1}{\rho_0}\nabla\delta p}+\frac{\eta}{\rho}\nabla^2\delta\mathbf{v}+\frac{1}{\rho_0}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\mathbf{v})$$

$$-\frac{1}{\rho_0}\left\{\left(\frac{\partial p}{\partial \rho}\right)_s\nabla\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\nabla\delta s\right\}=-\frac{c^2}{\rho_0}\nabla\delta\rho-\rho_0\left(\frac{\partial T}{\partial \rho}\right)_s\nabla\delta s$$

Digression -- from the first law of thermodynamics:

$$d\epsilon=Tds+\frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial}{\partial\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\right)_s=\left(\frac{\partial T}{\partial \rho}\right)_s \quad\Leftrightarrow\quad \left(\frac{\partial}{\partial s}\left(\frac{\partial\epsilon}{\partial \rho}\right)_s\right)_\rho=\left(\frac{\partial p/\rho^2}{\partial s}\right)_\rho\approx\frac{1}{\rho_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

$$\Rightarrow\frac{1}{\rho_0}\left(\frac{\partial p}{\partial s}\right)_\rho=\rho_0\left(\frac{\partial T}{\partial \rho}\right)_s$$

Linearized equations -- continued

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0$$

$$\rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0}\left(\left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2 \delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho \right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s}\right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

heat capacity at constant volume

Linearized equations -- continued

$$\rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0}\left(\left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2 \delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho \right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s}\right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \left(\gamma\kappa \nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho \right) \quad \text{where } \gamma \equiv \frac{c_p}{c_v}$$

Newton-Euler equations for viscous fluids – effects on sound
Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \delta\mathbf{v})$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta\mathbf{v}) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa \nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

Here: $\gamma = \frac{c_p}{c_v} \qquad \kappa = \frac{k_{th}}{c_p \rho_0}$

Linearized hydrodynamic equations

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa \nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

It can be shown that

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{Tc^2\beta}{\rho c_p} \approx \frac{T_0 c^2 \beta}{\rho_0 c_p} \qquad \text{where} \quad \beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \quad \text{(thermal expansion)}$$

Let $\quad \delta \mathbf{v} \equiv \delta \mathbf{v}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta\rho \equiv \delta\rho_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta s \equiv \delta s_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Linearized hydrodynamic equations; plane wave solutions:

$$\omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0 \beta c^2}{c_p}\delta s_0 \mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta \mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega \delta\rho_0 - \rho_0 \; \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega \delta s_0 = -i\gamma\kappa k^2 \delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2 \delta\rho_0$$

In the absense of thermal expansion, $\quad \beta = 0$

$$\omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta \mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega \delta\rho_0 - \rho_0 \; \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega \delta s_0 = -i\gamma\kappa k^2 \delta s_0$$

➔Entropy and mechanical modes are independent

In the absense of thermal expansion, $\quad \beta = 0$ -- continued:

$$\delta \mathbf{v} \equiv \delta \mathbf{v}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta\rho \equiv \delta\rho_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta s \equiv \delta s_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta \mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega \delta\rho_0 - \rho_0 \; \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega \delta s_0 = -i\gamma\kappa k^2 \delta s_0$$

Entropy wave: $\quad \delta s = \delta s_0 e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\lambda} e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\lambda - \omega t)} \quad$ where $\quad \lambda \equiv \sqrt{\frac{2\gamma\kappa}{\omega}}$

Density wave: $\quad \delta\rho_0 = \; \mathbf{k}\cdot\delta\mathbf{v}_0 \; \frac{\rho_0}{\omega} \qquad \delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} e^{i(\omega/c)(\hat{\mathbf{k}}\cdot\mathbf{r} - ct)}$

$$k = \frac{\omega}{c}\left(1 - i\frac{\omega}{c^2\rho_0}\left(\zeta + \frac{3}{4}\eta\right)\right)^{-1/2} \equiv \frac{\omega}{c} + i\alpha \approx \frac{\omega}{c} + i\frac{\omega^2}{2c^3\rho_0}\left(\zeta + \frac{3}{4}\eta\right)$$

Transverse wave: $\quad \delta \mathbf{v}_{0\perp} \times \mathbf{k} \neq 0 \qquad \delta \mathbf{v}_{\perp} = \delta \mathbf{v}_{0\perp} e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - \omega t)}$

where $\quad \delta \equiv \sqrt{\frac{2\eta}{\omega\rho_0}}$

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Transverse modes $(\delta\mathbf{v}\cdot\mathbf{k}=0)$:

$$\delta\rho_0 = 0 \qquad \delta s_0 = 0$$

$$\left(\omega + \frac{i\eta k^2}{\rho_0}\right)\left(\delta\mathbf{v}\times\mathbf{k}\right) = 0 \qquad k = \pm\left(\frac{i\omega\rho_0}{\eta}\right)^{1/2}$$

$$\delta\mathbf{v}_\perp = \delta\mathbf{v}_{0\perp}e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\delta-\omega t)} \qquad \text{where} \quad \delta \equiv \sqrt{\frac{2\eta}{\omega\rho_0}}$$

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k}\neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0\beta c^2k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Linearized hydrodynamic equations; full plane wave solutions:

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k}\neq 0)$:

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$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $k = \frac{\omega}{c} + i\alpha$

where $\alpha \approx \frac{\omega^2}{2c^3\rho_0}\left(\frac{4}{3}\eta + \zeta\right) + \frac{\kappa T_0\beta^2\omega^2}{2c_p c}$

$$\delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}}e^{i(\omega/c)\left(\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}$$