PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 37

Review

1. **Mathematical methods**
2. **Classical mechanics concepts**
3. **Course evaluation**

21	Fri, 10/19/2018	Chap. 7	Laplace transforms; Contour integrals	#13	10/24/2018
22	Mon, 10/22/2018	Chap. 7	Contour integrals		
23	Wed, 10/24/2018	Chap. 5	Rigid body motion	#14	10/26/2018
24	Fri, 10/26/2018	Chap. 5	Rigid body motion	#15	10/31/2018
25	Mon, 10/29/2018	Chap. 8	Mechanics of elastic membranes	#16	11/02/2018
26	Wed, 10/31/2018	Chap. 9	Mechanics of three dimensional fluids		
27	Fri, 11/02/2018	Chap. 9	Mechanics of fluids	#17	11/07/2018
28	Mon, 11/05/2018	Chap. 9	Sound waves	Project topic	
29	Wed, 11/07/2018	Chap. 9	Sound waves	#18	11/12/2018
30	Fri, 11/09/2018	Chap. 9	Linear and non-linear sound		
31	Mon, 11/12/2018	Chap. 10	Surface waves	#19	11/16/2018
32	Wed, 11/14/2018	Chap. 10	Surface waves -- nonlinear effects		
33	Fri, 11/16/2018	Chap. 11	Heat conductivity	#20	11/26/2018
34	Mon, 11/19/2018	Chap. 13	Elastic media		
	Wed, 11/21/2018	No class	Thanksgiving holiday		
	Fri, 11/23/2018	No class	Thanksgiving holiday		
35	Mon, 11/26/2018	Chap. 12	Viscous fluids		
36	Wed, 11/28/2018	Chap. 12	Viscous fluids		
37	Fri, 11/30/2018		Review		
	Mon, 12/03/2018		Presentations I		
	Wed, 12/05/2018		Presentations II		
	Fri, 12/07/2018		Presentations III		

Comments on presentation schedule:

PHY 711 Presentation Schedule for Fall 2018

Monday December 3, 2018

	Name	Topic
10-10:25	Leda Gao	Foucoult Pendulum
10:25-10:50		

Wednesday December 5, 2018

	Name	Topic
10-10:15	Dizhou Wu	Three Body
10:15-10:30	Eric Grotzke	TBD
10:30-10:45	Daniel Vickers	Inelastic Collisions in Interstellar Medium

Friday December 7, 2018

	Name	Topic
10-10:15	Lindsey Gray	Molecular vibrations
10:15-10:30	Ryan Sullivan	Chaotic Pendula
10:30-10:45	Shohreh	Vibrational modes in two dimensions

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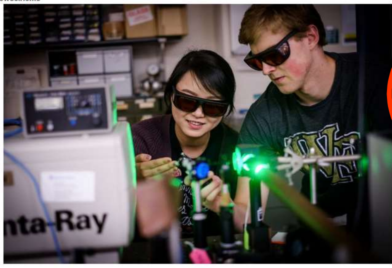

Events

Masters Defense: "Assessing Mutations in a Zinc Finger Domain of NF-KB Essential Modulator (Nemo) Using Molecular Dynamics Simulations" – Friday, November 30, 2018

Terra Dwayne Colvin, Jr., Masters Candidate, Public Presentation in Olin Physical Laboratory, Room #107 Friday, November 30, 2018, at 2:30 PM Fred Salsbury, PhD, Advisor The defense will follow. ABSTRACT …

Colloquium: "Probing electronic and magnetic interactions at complex-oxide interfaces" – Wednesday, November 28, 2018, 4:00 PM

Professor Divine Kumah, Department of Physics, North Carolina State University George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, November 28, 2018, at 4:00 PM There will be a reception …

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Comment from Lecture 36

$$\nabla^4 f(r)=0 \qquad f(r)=C_1r^2+C_2r+C_3+\frac{C_4}{r}$$

Some details:

$$\nabla^4 f(r)=\nabla^2\left(\nabla^2 f(r)\right)=0$$

$$\nabla^2 f(r)=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}f(r)=0 \qquad \Rightarrow f(r)=C_3+\frac{C_4}{r}$$

Now suppose: $\nabla^2 f(r)=g(r)$

$$\nabla^2 g(r)=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}g(r)=0 \qquad \Rightarrow g(r)=C_1+\frac{C_2}{r}$$

$$\nabla^2 f(r)=g(r)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}f(r)=C_1+\frac{C_2}{r} \qquad \Rightarrow f(r)=\frac{C_1r^2}{6}+\frac{C_2r}{2}+C_3+\frac{C_4}{r}$$

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Review of mathematical methods

Some useful identities for vectors and vector operators

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi=0$$

$$\nabla\cdot(\nabla\times\mathbf{a})=0$$

$$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$$

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Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

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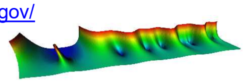


NIST Digital Library of Mathematical Functions

Project News

2018-09-15 *DLMF Update; Version 1.0.20*
2018-06-22 *DLMF Update; Version 1.0.19*
2018-06-22 *Philip J. Davis, A&S Author, dies at age 95*
2018-03-27 *DLMF Update; Version 1.0.18*
More news

Example – special functions

10 Bessel Functions
Bessel and Hankel Functions

10.1 Special Notation 10.3 Graphics

§10.2 Definitions

Contents

§10.2(i) Bessel's Equation

10.2.1 $$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0.$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$ of rank 1; compare §§2.7(i) and 2.7(ii).

§10.2(ii) Standard Solutions

Bessel Function of the First Kind

10.2.2 $$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu + k + 1)}.$$

This solution of (10.2.1) is an analytic function of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The *principal branch* of $J_\nu(z)$ corresponds to the principal value of $\left(\tfrac{1}{2}z\right)^\nu$ (§4.2(iv)) and is analytic in the z-plane cut along the interval $(-\infty, 0]$.

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

➔A closed integral of an analytic function is zero.

However:

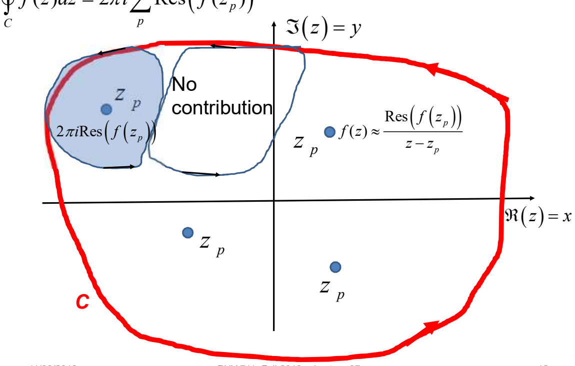
Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n}dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n}\int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Contour integration methods --

$$\oint_C f(z)dz = 2\pi i \sum_p \text{Res}(f(z_p))$$



General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{g(z)}{(z-z_p)^m} \underset{z \to z_p}{\equiv} \frac{\text{Res}(f(z_p))}{z-z_p}$

Since $g(z)$ is analytic near z_p, we can make a Taylor exansion about z_p :

$$g(z) \approx g(z_p) + (z-z_p)\frac{dg(z_p)}{dz} + \ldots + \frac{(z-z_p)^{m-1}}{(m-1)!}\frac{d^{m-1}g(z_p)}{dz^{m-1}} + \ldots$$

$$\Rightarrow \text{Res}(f(z_p)) = \lim_{z \to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left((z-z_p)^m f(z)\right)}{dz^{m-1}}\right\}$$

Fourier transforms --

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform :

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check :

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'}\right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt'\, f(t')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t'-t)}\right) = \int_{-\infty}^{\infty} dt'\, f(t')\, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = 2\pi\int_{-\infty}^{\infty} d\omega\, (F(\omega))^* F(\omega)$$

Check: $\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = \int_{-\infty}^{\infty} dt\left(\left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t}\right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t}\right)$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega'-\omega)$$

$$= 2\pi\int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

Doubly discrete Fourier Transforms

Doubly periodic functions

$$\omega \rightarrow \frac{2\pi\nu}{T} \qquad\qquad t \rightarrow \frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu = \frac{1}{2N+1}\sum_{\nu=-N}^{N} \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^{N} \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

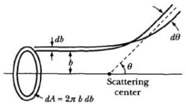
➔Fast Fourier Transforms (FFT)

Mechanics topics

- Scattering theory
- Lagrangian mechanics
- Hamiltonian mechanics
- Liouville theorem
- Rigid body motion
- Normal modes of oscillation about equilibrium
- Wave motion
- Fluid mechanics (ideal or including viscosity; linear and nonlinear)
- Heat conduction
- Mechanics of elastic response

Scattering theory

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



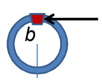


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Lagrangian mechanics

Given the Lagrangian function: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

For the case that there both mechanical and electromagnetic contributions in terms of electric and magnetic fields:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

$$L = T - U_{\text{mech}} - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

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Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

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Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space : $D = D(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem : $\frac{dD}{dt} = 0$

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Rigid body motion

Moment of inertia tensor :

$$\ddot{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

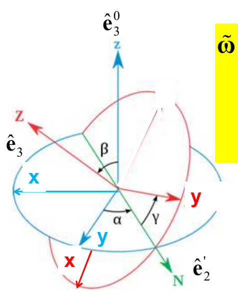
In a reference frame attached to the object, there are 3 moments of inertia and 3 distinct principal axes

Representation of rotational kinetic energy:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$
$$= \frac{1}{2} I_1 \left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma \right]^2$$
$$+ \frac{1}{2} I_2 \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma \right]^2$$
$$+ \frac{1}{2} I_3 \left[\dot{\alpha}\cos\beta + \dot{\gamma} \right]^2$$

Euler's transformation between body fixed and inertial reference frames

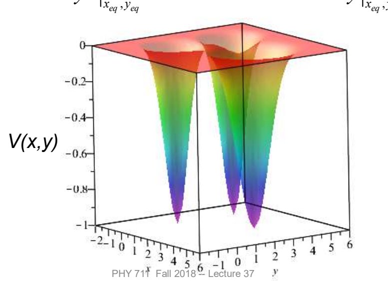
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



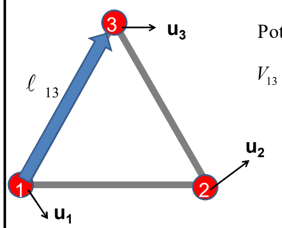
$$\tilde{\boldsymbol{\omega}} = \left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma \right] \hat{\mathbf{e}}_1 + \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma \right] \hat{\mathbf{e}}_2 + \left[\dot{\alpha}\cos\beta + \dot{\gamma} \right] \hat{\mathbf{e}}_3$$

Normal modes of vibration -- potential in 2 and more dimensions

$$V(x,y) \approx V(x_{eq},y_{eq}) + \tfrac{1}{2}\left(x - x_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_{eq},y_{eq}} + \tfrac{1}{2}\left(y - y_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial y^2}\right|_{x_{eq},y_{eq}} + \left(x - x_{eq}\right)\left(y - y_{eq}\right) \left.\frac{\partial^2 V}{\partial x \partial y}\right|_{x_{eq},y_{eq}}$$



Example – normal modes of a system with the symmetry of an equilateral triangle -- continued



$$\boldsymbol{\ell}_{13} = |\boldsymbol{\ell}_{13}|\left(\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}\right)$$

Potential contribution for spring 13:

$$V_{13} = \frac{1}{2}k\left(|\boldsymbol{\ell}_{13} + \mathbf{u}_3 - \mathbf{u}_1| - |\boldsymbol{\ell}_{13}|\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{13}\cdot(\mathbf{u}_3 - \mathbf{u}_1)}{|\boldsymbol{\ell}_{13}|}\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contributions: $V = V_{12} + V_{13} + V_{23}$

$$\approx \frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{12}\cdot(\mathbf{u}_2 - \mathbf{u}_1)}{|\boldsymbol{\ell}_{12}|}\right)^2 + \frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{13}\cdot(\mathbf{u}_3 - \mathbf{u}_1)}{|\boldsymbol{\ell}_{13}|}\right)^2$$

$$+\frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{23}\cdot(\mathbf{u}_3 - \mathbf{u}_2)}{|\boldsymbol{\ell}_{23}|}\right)^2$$

$$\approx \frac{1}{2}k(u_{x2} - u_{x1})^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}(u_{x2} - u_{x3}) - \frac{\sqrt{3}}{2}(u_{y2} - u_{y3})\right)^2$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

$$\frac{k}{m}\begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} \\ -1 & \frac{5}{4} & -\frac{1}{4} & 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 \\ \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & \frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & -\frac{3}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = \omega^2\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$$

Discrete particle interactions → continuous media →
The wave equation

Initial value solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Mechanical motion of fluids

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

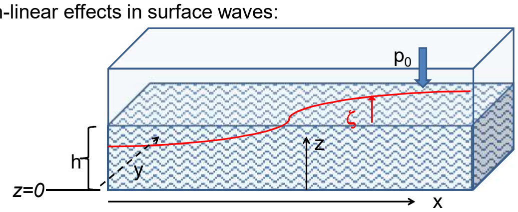
Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Viscosity contributions

Fluid mechanics of incompressible fluid plus surface

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$