PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 3:
Textbook reading: Chapter 1

1. Introduction to scattering theory
2. Center of mass reference fame and its relationship to "laboratory" reference frame
3. Evaluation of the differential scattering cross section

PHY 711 Classical Mechanics and Mathematical Methods

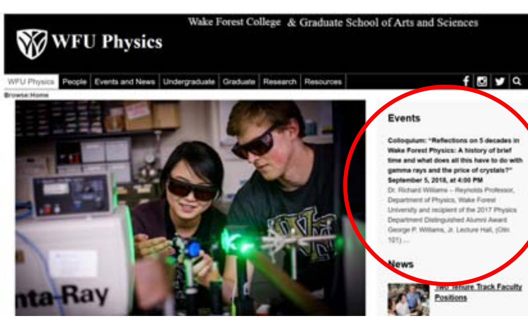
MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f18phy711/

Instructor: Natalie Holzwarth	Phone:758-5510	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
2	Wed, 8/29/2018	No class			
3	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
4	Mon, 9/03/2018	Chap. 1	Scattering theory		



PHY 711 -- Assignment #2

Aug. 31, 2018

Read Chapter 1 in **Fetter & Walecka**.

1. In class, we "derived" the differential cross section for the scattering of two hard spheres of mutual radius D in the center of mass frame. Analyze this system directly in the laboratory frame in which the target of mass m_{target} is initially at rest and the scattering particle has mass m and initial velocity $\mathbf{V}_0$. Consider the following two cases, finding the differential cross section as a function of lab angle for each (You may wish to check your answers using the center of mass expressions.)
   a. $m << m_{target}$
   b. $m = m_{target}$

Note: Part b is hard. Do the best you can.

Scattering theory:

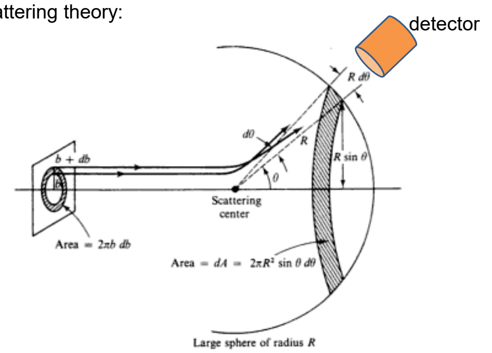
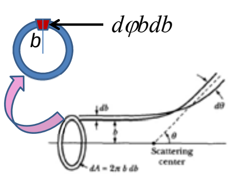


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

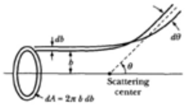
$= $ Area of incident beam that is scattered into detector at angle θ



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



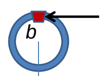
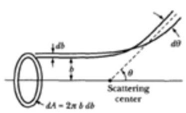


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Simple example – collision of hard spheres



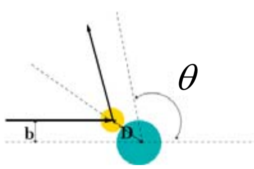
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

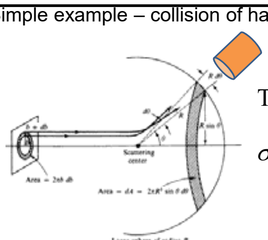
$$b(\theta) = ?$$

$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$



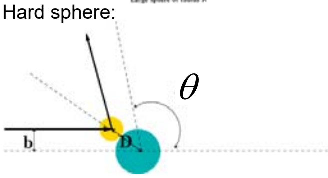
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

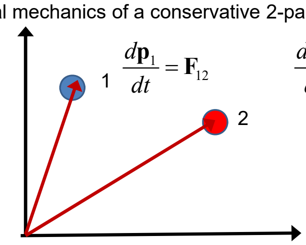
Hard sphere:



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

Relationship of scattering cross-section to particle interactions -- Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \quad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \le a \\ 0 & r > a \end{cases}$

Coulomb or gravitational: $V(r) = \frac{K}{r}$

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1 + m_2)\dot{\mathbf{R}}_{CM} = (m_1 + m_2)\mathbf{V}_{CM}$$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

Classical mechanics of a conservative 2-particle system -- continued

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu\left|\mathbf{v}_1-\mathbf{v}_2\right|^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12}\equiv\mathbf{r}_{12}\times\mu\mathbf{v}_{12}$$

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu v_{12}^2+\frac{L_{12}^2}{2\mu r_{12}^2}+V(r_{12})$$

Simpler notation:

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V(r)$$

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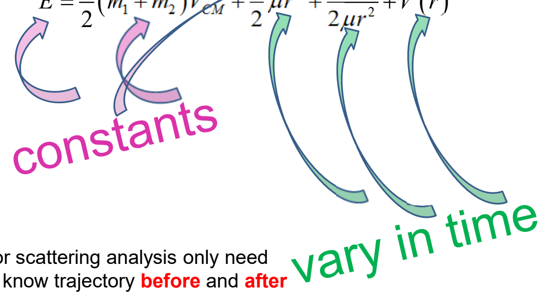
Simpler notation:

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V(r)$$



For scattering analysis only need to know trajectory **before** and **after** the collision.

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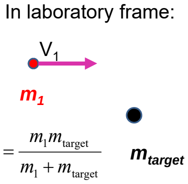
Note: The following analysis will be carried out in the center of mass frame of reference.

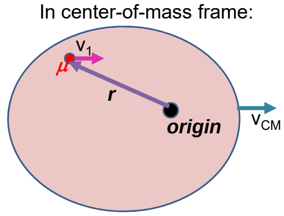
In laboratory frame:



$$\mu=\frac{m_1 m_{\text{target}}}{m_1+m_{\text{target}}}$$

$$\ell=|\mathbf{r}\times\mu\mathbf{v}_1|$$

In center-of-mass frame:



Also note: We are assuming that the interaction between particle and target $V(r)$ conserves energy and angular momentum.

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It is often convenient to analyze the scattering cross section in the center of mass reference frame.

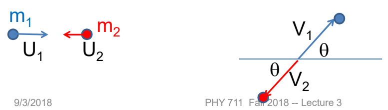
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before | After

Center of mass reference frame:

Before | After



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

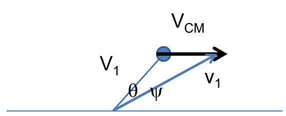
$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1+m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1+m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\psi = V_1 \sin\theta$$

$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

For elastic scattering

Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2 + \tfrac{1}{2}m_2U_2^2 + \tfrac{1}{2}(m_1 + m_2)V_{CM}^2$$
$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \tfrac{1}{2}(m_1 + m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1 + m_2\mathbf{U}_2 = 0 \qquad m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0$$

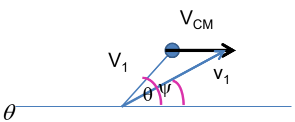
$$\mathbf{U}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM} \qquad \mathbf{U}_2 = -\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U}_1| = |\mathbf{V}_1| \quad \text{and} \quad |\mathbf{U}_2| = |\mathbf{V}_2| = |\mathbf{V}_{CM}|$$

$$\text{Also note that: } m_1|\mathbf{U}_1| = m_2|\mathbf{U}_2|$$

$$\text{So that: } V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1\cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\text{Also: } \cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}} = \left|\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}\right| = \left|\frac{d\cos\theta}{d\cos\psi}\right|$$

Using:

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2(m_1/m_2)\cos\theta + (m_1/m_2)^2}}$$

$$\left|\frac{d\cos\psi}{d\cos\theta}\right| = \frac{(m_1/m_2)\cos\theta + 1}{\left(1 + 2(m_1/m_2)\cos\theta + (m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where: }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where: }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case: }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+1}\quad\Rightarrow\psi=\frac{\theta}{2}$$

$$\text{note that }\quad 0\leq\psi\leq\frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Summary --

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

For elastic scattering

$$\text{where: }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

where : $\tan\psi = \dfrac{\sin\theta}{\cos\theta + m_1/m_2}$

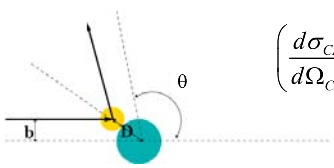
Example: suppose $m_1 = m_2$

In this case : $\tan\psi = \dfrac{\sin\theta}{\cos\theta+1} \quad \Rightarrow \psi = \dfrac{\theta}{2}$

note that $0 \le \psi \le \dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Example of hard spheres



$$\left(\frac{d\sigma_{CM}}{d\Omega_{CM}}\right) = \frac{D^2}{4}$$

Cross section in lab frame when $m_1 = m_2$

$\tan\psi = \dfrac{\sin\theta}{\cos\theta+1} \quad \Rightarrow \psi = \dfrac{\theta}{2} \quad \Rightarrow$ note that $0 \le \psi \le \dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi = D^2\cos\psi$$

Hard sphere example – continued

$m_1 = m_2$

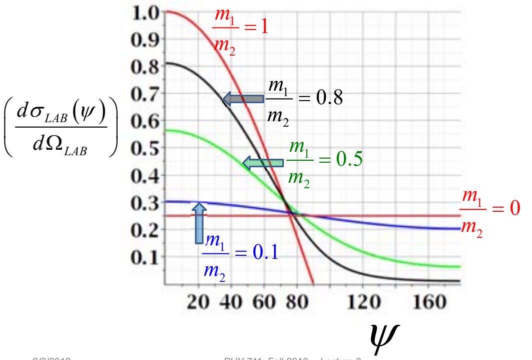
Center of mass frame | Lab frame

$$\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right) = \frac{D^2}{4} \qquad \left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = D^2\cos\psi \quad \psi = \frac{\theta}{2}$$

$$\int \frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}d\Omega_{CM} = \qquad \int \frac{d\sigma_{lab}(\psi)}{d\Omega_{lab}}d\Omega_{lab} =$$

$$\frac{D^2}{4}4\pi = \pi D^2 \qquad 2\pi D^2\int_0^{\pi/2}\cos\psi\ \sin\psi d\psi = \pi D^2$$

Scattering cross section for hard sphere in lab frame for various mass ratios:



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For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \psi(\theta)$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

Maple syntax:

```
> plot( { [psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
   = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
   sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
   thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
   orange])
```

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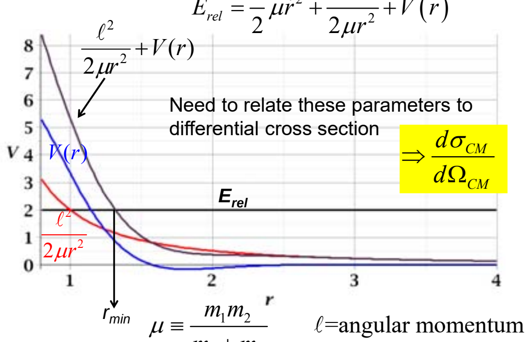
For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section $\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

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Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \le D \\ 0 & r > D \end{cases}$ X

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ←

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$