

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

**Plan for Lecture 4:
Reading: Chapter 1 F&W**

- 1. Summary of previous discussion of scattering theory; transformation between lab and center of mass frames**
- 2. Scattering theory in the center of mass frame; Calculation of the scattering cross section**
- 3. Cross section for Rutherford scattering**

Physics colloquium today --

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Events

Colloquium: "Reflections on 5 decades in Wake Forest Physics: What does it take in time and what does all this have to do with gamma rays and the price of crystals?" September 3, 2018, 4:30pm
Dr. Thomas J. Williams - Reynolds Professor, Department of Physics, Wake Forest University and recipient of the 2017 Physics Department Distinguished Alumni Award
George P. Williams, Jr., Lecture Hall, (Office)

Colloquium: "Please to Diagnosis: How to design, develop and evaluate a medical Image analysis system?" September 12, 2018, 4PM

PHY 711 Classical Mechanics and Mathematical Methods				
MWF 10 AM-10:50 AM OPL 103 http://www.wfu.edu/~natalie/f18phy711/				
Instructor:	Natalie Holzwarth	Phone:	758-5510	Office: 300 OPL e-mail: natalie@wfu.edu
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<h2>Course schedule</h2>				
(Preliminary schedule -- subject to frequent adjustment.)				
Date	F&W Reading	Topic	Assignment Due	
1 Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
Wed, 8/29/2018	No class			
2 Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3 Mon, 9/03/2018	Chap. 1	Scattering theory		
4 Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5 Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems		

PHY 711 -- Assignment #2

Aug. 31, 2018

Read Chapter 1 in Fetter & Walecka

1. In class, we "derived" the differential cross section for the scattering of two hard spheres of mutual radius D in the center of mass frame. Analyse this system directly in the laboratory frame in which the target of mass m_{target} is initially at rest and the scattering particle has mass m and initial velocity \mathbf{v}_0 . Consider the following two cases, finding the differential cross section as a function of lab angle for each. (You may wish to check your answers using the center of mass expressions.)

- $m \ll m_{\text{target}}$
- $m = m_{\text{target}}$

PHY 711 -- Assignment #3

Sept. 5, 2018

Continue reading Chapter 1 in Fetter & Walecka

1. Work Problem #1.16 at the end of Chapter 1 in Fetter and Walecka.

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Scattering theory:

detector

Scattering center

Large sphere of radius R

Area = $2\pi b db$

Area = $dA = 2\pi R^2 \sin \theta d\theta$

Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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Differential cross section

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector

at angle θ

$$d\sigma = d\phi b db$$

$$d\Omega = d\phi \sin \theta d\theta$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

Figure from Marion & Thornton, Classical Dynamics

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Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations



Figure from Marion & Thornton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin\theta d\theta} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

Note: We are assuming that the process is isotropic in ϕ

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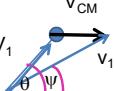
Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)

Lab (θ) vs CM (ψ)

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2 \cos\theta + (m_1/m_2)^2}}$$



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Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \left| \frac{d\cos\theta}{d\cos\psi} \right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{(1 + 2m_1/m_2 \cos\theta + (m_1/m_2)^2)^{3/2}}{(m_1/m_2) \cos\theta + 1}$$

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Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{\left(1 + 2m_1/m_2 \cos \theta + (m_1/m_2)^2 \right)^{3/2}}{(m_1/m_2) \cos \theta + 1}$$

$$\text{where : } \tan \psi = \frac{\sin \theta}{\cos \theta + m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case : } \tan \psi = \frac{\sin \theta}{\cos \theta + 1} \Rightarrow \psi = \frac{\theta}{2}$$

note that $0 \leq \psi \leq \frac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}} \right) \cdot 4 \cos \psi$$

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Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

$$\text{Hard sphere: } V(r) = \begin{cases} \infty & r \leq D \\ 0 & r > D \end{cases}$$

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ↶

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

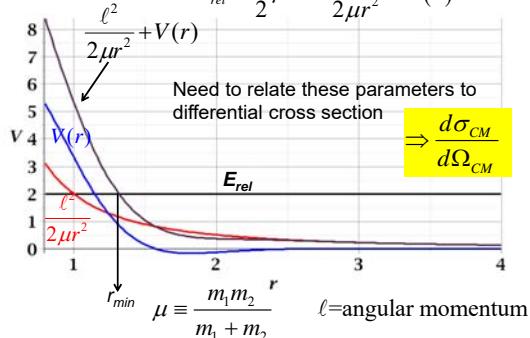
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For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



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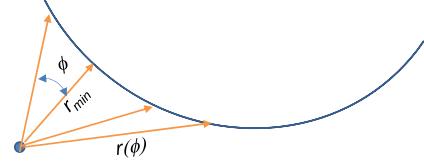
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$$E_{rel} = \frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame
 $r(\phi)$

→ Need to find an equation for $r(\phi)$



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Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi} \frac{d\varphi}{dt} = \frac{dr}{d\varphi} \frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2 \left(\frac{d\varphi}{dt} \right)$

$$\Rightarrow E = \frac{1}{2} \mu \left(\frac{dr}{d\varphi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

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Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

$$\text{From: } E = \frac{1}{2} \mu \left(\frac{dr}{d\varphi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right) \left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

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The diagram illustrates a rotating fluid element in a cylindrical coordinate system \$(r, \theta, z)\$. A horizontal dashed line represents the \$z\$-axis. A curved black line represents the trajectory of a fluid element, starting from a point at radius \$r\$ and angle \$\phi_m\$ and curving upwards towards a point at radius \$r_{max}\$ and angle \$\pi - \phi_m\$. A vertical dashed line extends from the center of rotation. A horizontal arrow labeled \$V_\infty\$ points to the right, indicating the free-stream velocity. The angle \$\theta\$ is shown between the radial direction and the trajectory curve.

Special values at large separation ($r \rightarrow \infty$):

$$\ell = \mu |\mathbf{r} \times \mathbf{v}|_{r \rightarrow \infty} = \mu v_\infty b$$

$$E = \frac{1}{2} \mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E b}$$

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When the dust clears:

$$d\varphi = dr \left(-\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2ur^2} - V(r) \right)}} \right)$$

$$d\varphi = dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\Rightarrow \varphi_{\max}(b, E) = \varphi(r \rightarrow \infty) - \varphi(r = r_{\min})$$

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$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

$$1 - \frac{b^2}{r_{\perp}^2} - \frac{V(r_{\min})}{E} = 0$$

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Relationship between ϕ_{\max} and θ :



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

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$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - b^2/r^2 - V(r)/E}} \right)$$

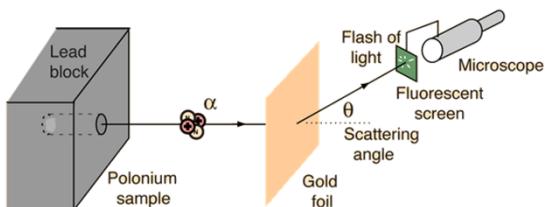
$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

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Example: Diagram of Rutherford scattering experiment
<http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html>



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Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\begin{aligned} \frac{V(r)}{E} &\equiv \frac{\kappa}{r} & 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} &= 0 \\ \frac{1}{r_{\min}} &= \frac{1}{b} \left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right) \\ \theta &= -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \kappa u}} \right) = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right) \end{aligned}$$

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Rutherford scattering continued :

$$\theta = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

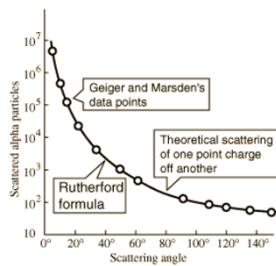
$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

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$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: <http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3>

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Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\text{Au}} e^2}{8\pi\epsilon_0\mu v_\infty^2} = \frac{Z_\alpha Z_{\text{Au}} e^2}{16\pi\epsilon_0 E_{\text{rel}}}$$

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Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where r_{\min} is found from

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

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