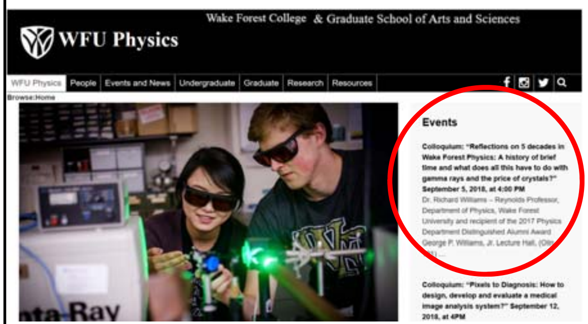
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 4:
Reading: Chapter 1 F&W

1. Summary of previous discussion of scattering theory; transformation between lab and center of mass frames
2. Scattering theory in the center of mass frame; Calculation of the scattering cross section
3. Cross section for Rutherford scattering

Physics colloquium today --



PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f18phy711/

Instructor: Natalie Holzwarth	Phone:758-5510	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems		

PHY 711 -- Assignment #2

Aug. 31, 2018

Read Chapter 1 in **Fetter & Walecka**.

1. In class, we "derived" the differential cross section for the scattering of two hard spheres of mutual radius D in the center of mass frame. Analyze this system directly in the laboratory frame in which the target of mass m_{target} is initially at rest and the scattering particle has mass m and initial velocity $\mathbf{V}_0$. Consider the following two cases, finding the differential cross section as a function of lab angle for each (You may wish to check your answers using the center of mass expressions.)
   a. $m << m_{target}$
   b. $m = m_{target}$

PHY 711 -- Assignment #3

Sept. 5, 2018

Continue reading Chapter 1 in **Fetter & Walecka**.

1. Work Problem #1.16 at the end of Chapter 1 in **Fetter and Walecka**.

Scattering theory:

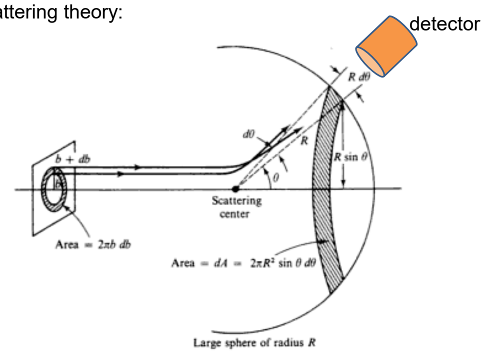
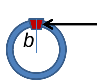


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$= \text{Area of incident beam that is scattered into detector at angle } \theta$



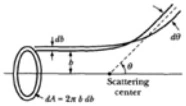
$$d\sigma = d\phi b db$$

$$d\Omega = d\phi \sin\theta d\theta$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations



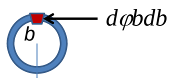
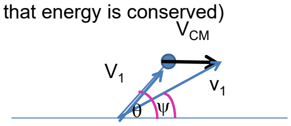


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)



$$\text{Lab } (\theta) \quad \text{vs} \quad \text{CM } (\psi)$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+1}\quad\Rightarrow\psi=\frac{\theta}{2}$$

$$\text{note that}\quad 0\le\psi\le\frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r)$

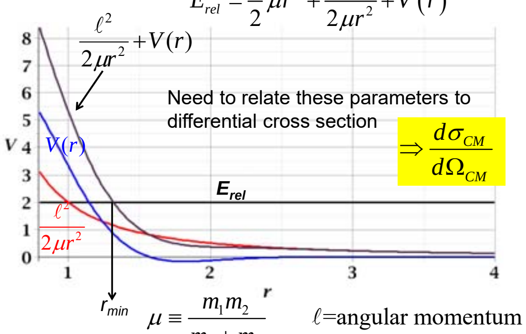
Hard sphere: $V(r)=\begin{cases}\infty & r\le D\\ 0 & r>D\end{cases}$ X

Coulomb or gravitational: $V(r)=\dfrac{K}{r}$ ←

Lennard-Jones: $V(r)=\dfrac{A}{r^{12}}-\dfrac{B}{r^6}$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel}=\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V(r)$$

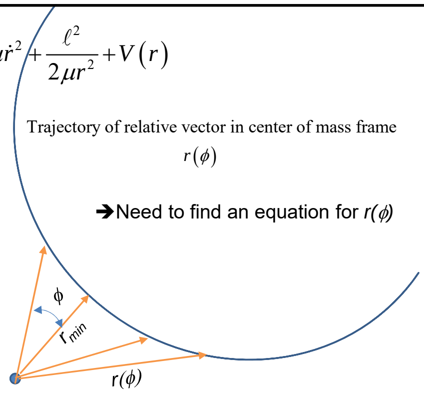


$\mu\equiv\dfrac{m_1m_2}{m_1+m_2}$ ℓ=angular momentum

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame $r(\phi)$

➔Need to find an equation for $r(\phi)$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$r(t) \Leftrightarrow r(\varphi)$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

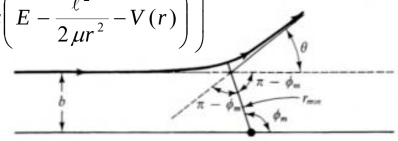
From: $E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

v_∞ →



Special values at large separation ($r \to \infty$):

$$\ell = \mu |\mathbf{r} \times \mathbf{v}|_{r\to\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

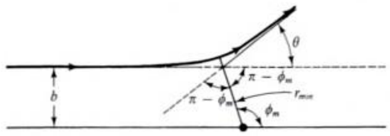
$$\Rightarrow \varphi_{\max}(b, E) = \varphi(r \to \infty) - \varphi(r = r_{\min})$$

$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

where :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Relationship between ϕ_{max} and θ:



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

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$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

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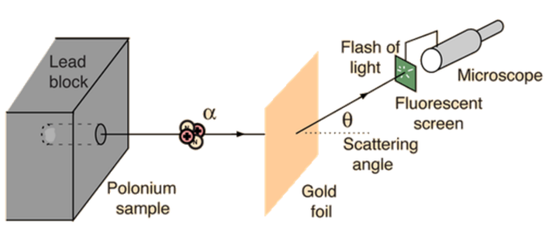
Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



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Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

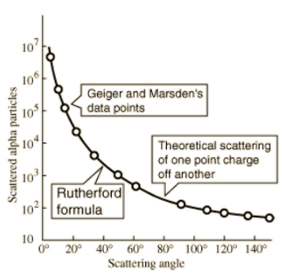
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\text{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\text{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

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Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\theta = -\pi + 2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where $r_{\min}$ is found from

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

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