PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 5:
Reading: Chapter 2 of Fetter & Walecka

Physics described in accelerated coordinate frames

1. **Linear acceleration**
2. **Angular acceleration**
3. **Foucault pendulum**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation		

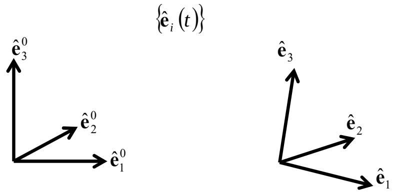
Fall 2018 Schedule
for N. A. W. Holzwarth

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00-10:00	Lecture Preparation/ Office Hours	Physics Research	Lecture Preparation/ Office Hours	Physics Research	Lecture Preparation/ Office Hours
10:00-11:00	Classical Mechanics PHY711		Classical Mechanics PHY711		Classical Mechanics PHY711
11:00-12:00	Office Hours		Office Hours		Office Hours
12:00-12:30	CEES Lunch Series		Physics Research		Physics Research
12:30-1:00				Condensed Matter Theory Journal Club	
1:00-1:30	Physics Research				
1:30-3:30				Physics Research	
3:30-5:00			Physics Colloquium		

Physics in accelerated reference frames

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\left\{\hat{\mathbf{e}}_i^0\right\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\left\{\hat{\mathbf{e}}_i(t)\right\}$



Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

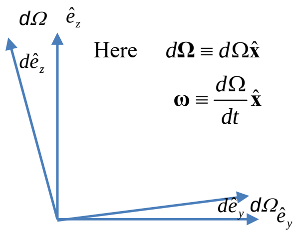
$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define : } \left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Properties of the frame motion (rotation only):



Here $d\boldsymbol{\Omega} \equiv d\Omega\hat{\mathbf{x}}$

$$\boldsymbol{\omega} \equiv \frac{d\Omega}{dt}\hat{\mathbf{x}}$$

$$d\hat{e}_y = d\Omega\hat{e}_z$$

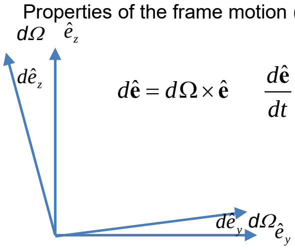
$$d\hat{e}_z = -d\Omega\hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\boldsymbol{\Omega}\times\hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\boldsymbol{\Omega}}{dt}\times\hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{e}}$$

Properties of the frame motion (rotation only):



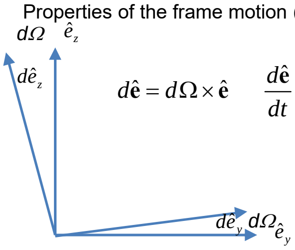
$$d\hat{\mathbf{e}} = d\Omega\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt}\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\Omega\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt}\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} = -d\Omega e_z\hat{\mathbf{e}}_y + d\Omega e_y\hat{\mathbf{e}}_z = d\Omega\hat{\mathbf{x}}\times\hat{\mathbf{e}}$$

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial}$

Newton's laws; Let **r** denote the position of particle of mass *m*:

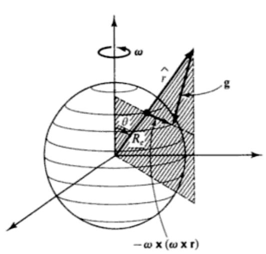
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

Coriolis force

Centrifugal force

Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3 \times 10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Earth's gravity

Support force

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

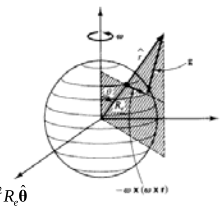
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



0.03 m/s^2

9.80 m/s^2

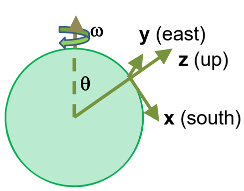
Foucault pendulum

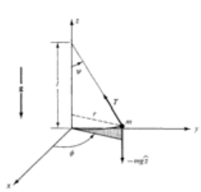
http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

Equation of motion on Earth's surface

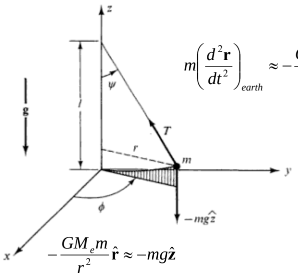
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + (\dot{x}\cos\theta + \dot{z}\sin\theta)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

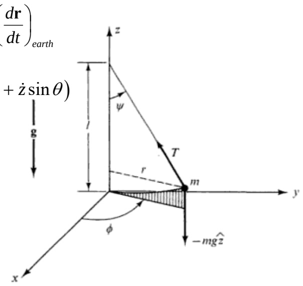
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\varphi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -T\sin\psi\sin\varphi - 2m\omega(\dot{x}\cos\theta + \dot{z}\sin\theta)$$

$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin\theta$$



Further approximation :

$\psi << 1;\quad \ddot{z} \approx 0;\quad T \approx mg$

$$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$$

Also note that :

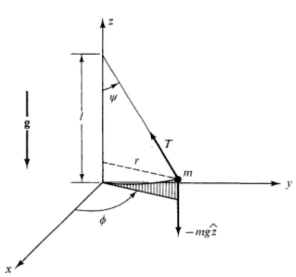
$$x \approx \ell\sin\psi\cos\phi$$

$$y \approx \ell\sin\psi\sin\phi$$

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$



Try to find a solution of the form :

$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

Denote $\quad \omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\,2} + \frac{g}{\ell}}$$

Foucault pendulum continued – coupled equations:

Solution continued :

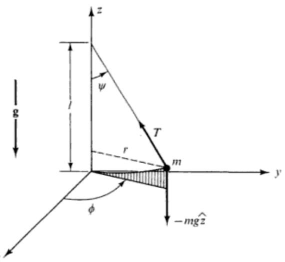
$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions :

$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$

Amplitude relationship : $X = iY$



General solution with complex amplitudes C and D :

$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$

$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$

General solution with complex amplitudes C and D :

$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$

$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}} \approx \omega_\perp \pm \sqrt{\frac{g}{\ell}}$$

since $\quad \omega_\perp \approx 7\times10^{-5}\cos\theta \;\; rad/s << \sqrt{\frac{g}{\ell}}$

Suppose : $\quad x(0) = X_0 \qquad y(0) = 0$

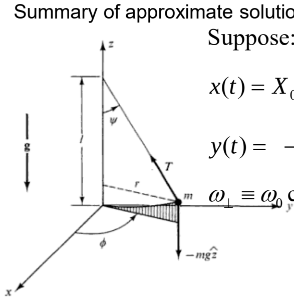
$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos(\omega_\perp t)$

$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin(\omega_\perp t)$

Note that

$\omega = \frac{2\pi}{24\cdot3600\text{ s}} = 7\times10^{-5}$ rad/sec

Summary of approximate solution for Foucault pendulum:

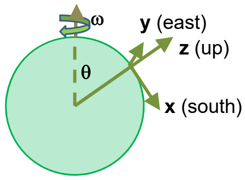


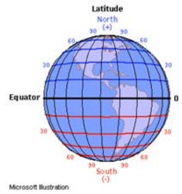
Suppose: $\quad x(0) = X_0 \qquad y(0) = 0$

$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos(\omega_\perp t)$

$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin(\omega_\perp t)$

$\omega_\perp \equiv \omega_0 \cos\theta$





$$\omega_\perp \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$