

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 6:

Start reading Chapter 3.17 –

- 1. Introduction to the calculus of variations**
- 2. Example problems**
- 3. Brachistochrone**

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

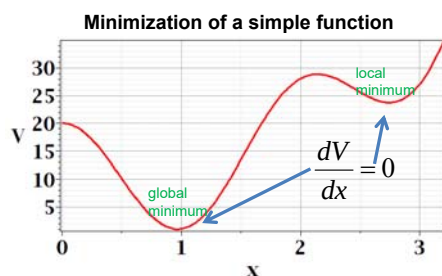
Date	F&W Reading	Topic	Assignment Due
1 Mon, 8/27/2018	Chap. 1	Introduction	#1 9/7/2018
Wed, 8/29/2018	No class		
2 Fri, 8/31/2018	Chap. 1	Scattering theory	#2 9/7/2018
3 Mon, 9/03/2018	Chap. 1	Scattering theory	
4 Wed, 9/05/2018	Chap. 1	Scattering theory	#3 9/10/2018
5 Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4 9/12/2018
6 Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5 9/12/2018
7 Wed, 9/12/2018	Chap. 3	Calculus of Variation	

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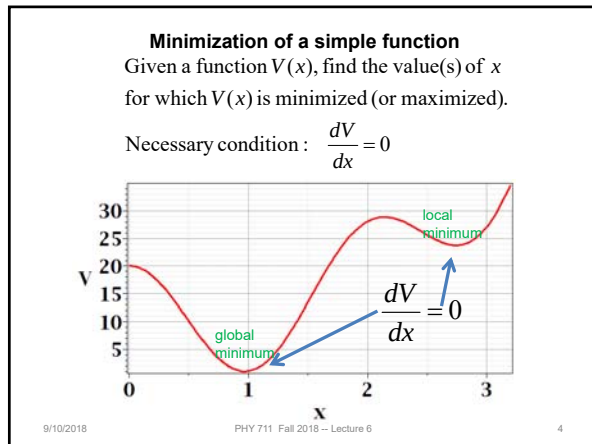
In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

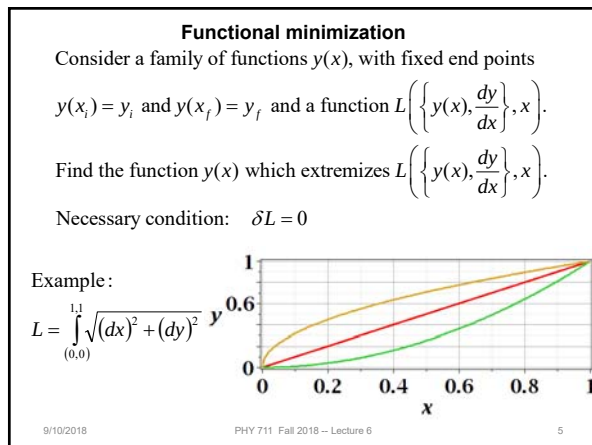


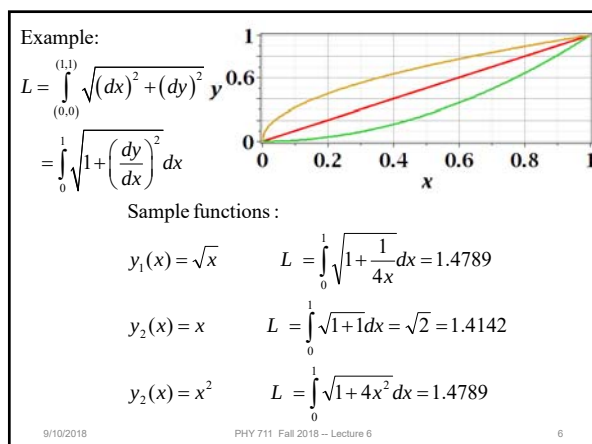
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Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$.

Necessary condition : $\delta L = 0$

At any x , let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx.$$

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After some derivations, we find

$$\begin{aligned} \delta L &= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx \\ &= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f \\ \Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] &= 0 \quad \text{for all } x_i \leq x \leq x_f \end{aligned}$$

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“Some” derivations --

Consider term

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] dx :$$

If $y(x)$ is a well-defined function, then $\delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} \delta y$

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] dx = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \frac{d}{dx} \delta y \right] dx$$

$$= \int_{x_i}^{x_f} \left[\frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] \delta y - \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta y \right] dx$$

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"Some" derivations (continued)--

$$\begin{aligned} & \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right) - \frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= 0 - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \end{aligned}$$

Euler-Lagrange equation:

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

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Example: End points -- $y(0) = 0; y(1) = 1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \Rightarrow f \left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0$$

$$\Rightarrow - \frac{d}{dx} \left(\frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = K \quad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1 - K^2}}$$

$$\Rightarrow y(x) = K'x + C \quad \mathbf{y(x) = x}$$

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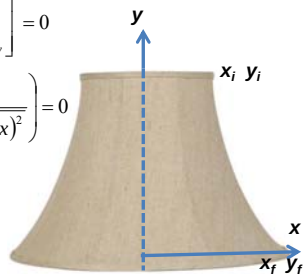
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Example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \Rightarrow f \left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) = x \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0$$

$$\Rightarrow - \frac{d}{dx} \left(\frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$



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$$-\frac{d}{dx} \left(\frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} \right) = 0$$

$$\frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} = K_1$$

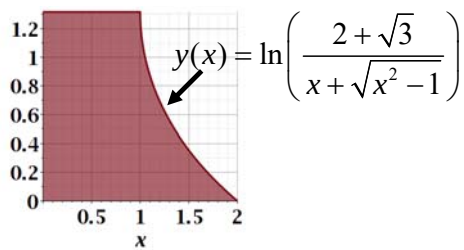
$$\frac{dy}{dx} = -\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}}$$

$$\Rightarrow y(x) = K_2 - K_1 \ln \left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

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Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$ that minimize the integral :

$$I = \int_0^1 \left(\left(\frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2 y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin(\sqrt{a}x)}{\sin(\sqrt{a})}$$

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Review: for $f\left(y(x), \frac{dy}{dx}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

Note that for $f\left(y(x), \frac{dy}{dx}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right) \quad \leftarrow \text{Alternate Euler-Lagrange equation}$$

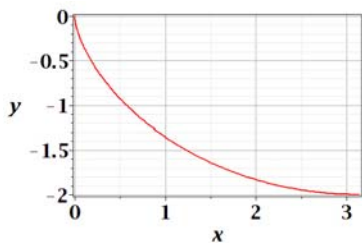
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Brachistochrone problem: (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight mg travels frictionlessly down a path of shape $y(x)$. What is the shape of the path $y(x)$ that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

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$$T = \int_{x_i, y_i}^{x_f, y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because } \frac{1}{2}mv^2 = -mgy$$

$$f\left(y(x), \frac{dy}{dx}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}} \quad \text{Note that for the original form of Euler-Lagrange equation:}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = 0 \quad \left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0,$$

differential equation is more complicated:

$$\frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx}\right)^2\right)}} \right) = 0 \quad \frac{1}{2} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx} \left(\frac{\frac{dy}{dx}}{\sqrt{-y \left(1 + \left(\frac{dy}{dx}\right)^2\right)}} \right) = 0$$

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$$f\left(y(x), \frac{dy}{dx}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1 + \left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \quad -y\left(1 + \left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

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$$-y\left(1 + \left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a \quad \text{Let } y = -2a \sin^2 \frac{\theta}{2} = a(\cos \theta - 1)$$

$$\frac{dy}{dx} = -\frac{\frac{2a}{-y} - 1}{\sqrt{-y}} \quad -\frac{dy}{\sqrt{-y}} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sqrt{2a \sin^2 \frac{\theta}{2} - 1}} = dx$$

$$-\frac{dy}{\sqrt{-y}} = dx \quad x = \int_0^\theta a(1 - \cos \theta') d\theta' = a(\theta - \sin \theta)$$

Parametric equations for Brachistochrone:

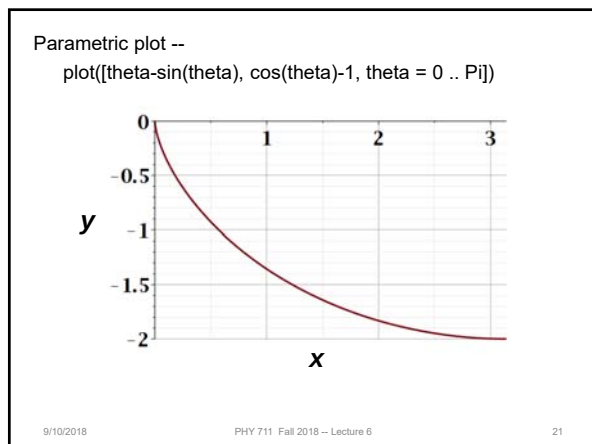
$$x = a(\theta - \sin \theta)$$

$$y = a(\cos \theta - 1)$$

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