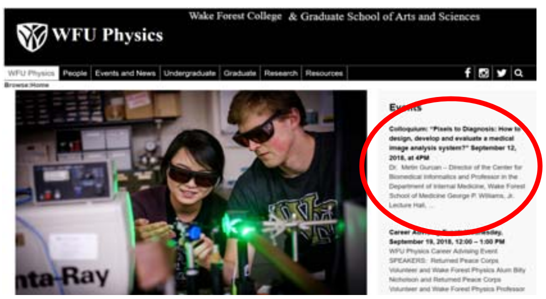
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 7:

Calculus of variation; Continue reading Chapt. 3

1. Brachistochrone problem
2. Calculus of variation with constraints
3. Application to classical mechanics



Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/14/2018
8	Fri, 9/14/2018	Chap. 3	Lagrangian Mechanics		

PHY 711 – Assignment #6

September 11, 2018

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.
   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$
   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$
   (c) Evaluate
   $$\frac{df}{dt}.$$
   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = e^{-t/\tau}.$$
   Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

Review: for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

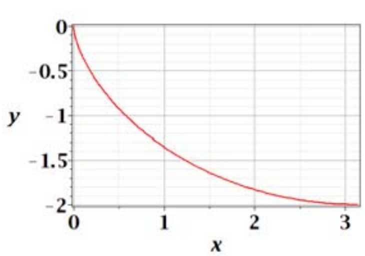
$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2* ?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = dx$$

Let $y = -2a\sin^2\frac{\theta}{2} = a(\cos\theta - 1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}} = dx$$

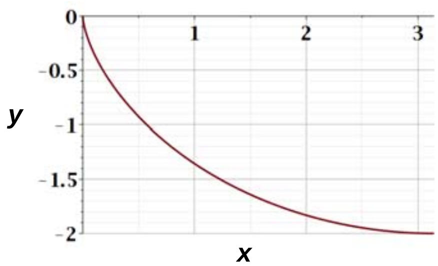
$$x = \int_0^\theta a(1-\cos\theta')d\theta' = a(\theta - \sin\theta)$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin\theta)$$
$$y = a(\cos\theta - 1)$$

Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$\left(\rho g y+\lambda\right)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$\left(\rho g y+\lambda\right)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : K, a, λ

Constraints : $y(x_1)=y_1$

$y(x_2)=y_2$

$$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I=\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx \qquad \delta I=0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right)-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right]=0$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics

$$x \rightarrow t \qquad \text{(time)}$$
$$y \rightarrow q \qquad \text{(generalized coordinate)}$$
$$f \rightarrow L \qquad \text{(Lagrangian)}$$
$$I \rightarrow A \qquad \text{(action)}$$

Denote: $\dot{q} \equiv \dfrac{dq}{dt}$

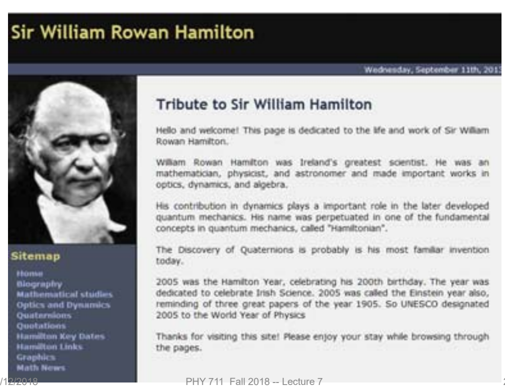
$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

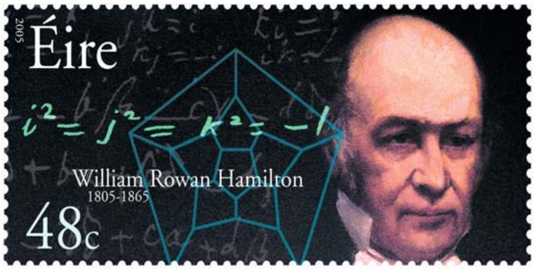
Application to particle dynamics

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a=-g.*

$$m\frac{d^2y}{dt^2} = -mg$$
$$y(t) = y_i + v_i t - \tfrac{1}{2}gt^2$$

http://www.hamilton2005.ie/





http://rjlipton.wordpress.com

Now consider the Lagrangian defined to be:

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy\right)dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt}m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2}gt^2$$

Check:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0,\ y_i = h \equiv \frac{1}{2} g T^2;\quad t_f = T,\ y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} g T^2 \left(1 - t/T\right) = h - \frac{1}{2} g T t$

$$y_2(t) = \frac{1}{2} g T^2 \left(1 - t^2/T^2\right) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 \left(1 - t^3/T^3\right) = h - \frac{1}{2} g t^3/T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$

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