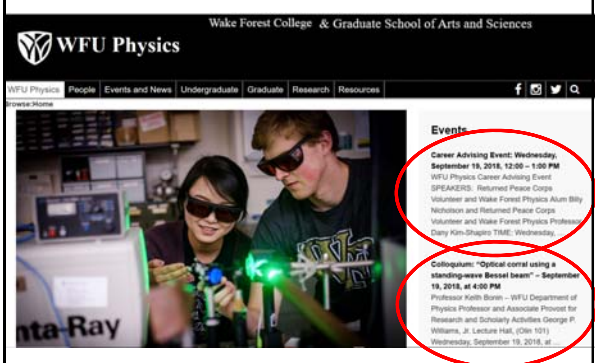
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 9:

Continue reading Chapter 3 & 6

1. **Summary & review**
2. **Lagrange's equations with constraints**



Comment on HW#7

Given $f(q(t),\dot{q}(t),t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial f}{\partial t}$$

$$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} = \frac{\partial^2 f}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial f}{\partial q} + \frac{\partial^2 f}{\partial \dot{q}\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial^2 f}{\partial \dot{q}\partial t}$$

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{q}} = \frac{\partial^2 f}{\partial q\partial \dot{q}}\dot{q} + \frac{\partial^2 f}{\partial \dot{q}\partial \dot{q}}\ddot{q} + \frac{\partial^2 f}{\partial t\partial \dot{q}}$$

Comment on HW #7 (continued)

$$\frac{\partial^2 f}{\partial q \partial \dot{q}} = \frac{\partial^2 f}{\partial \dot{q} \partial q}$$

Example: $f(q(t), \dot{q}(t), t) = q\dot{q}^2 t^2$

$$\frac{\partial^2 f}{\partial q \partial \dot{q}} = 2\dot{q}t^2 = \frac{\partial^2 f}{\partial \dot{q} \partial q}$$

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

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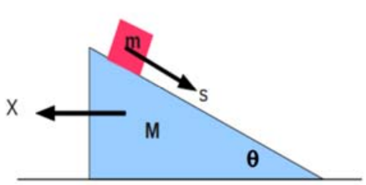
	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/27/2018	Chap. 1	Introduction	#1	9/7/2018
	Wed, 8/29/2018	No class			
2	Fri, 8/31/2018	Chap. 1	Scattering theory	#2	9/7/2018
3	Mon, 9/03/2018	Chap. 1	Scattering theory		
4	Wed, 9/05/2018	Chap. 1	Scattering theory	#3	9/10/2018
5	Fri, 9/07/2018	Chap. 2	Non-inertial coordinate systems	#4	9/12/2018
6	Mon, 9/10/2018	Chap. 3	Calculus of Variation	#5	9/12/2018
7	Wed, 9/12/2018	Chap. 3	Calculus of Variation	#6	9/17/2018
	Fri, 9/14/2018	No class	University closed due to weather.		
8	Mon, 9/17/2018	Chap. 3	Lagrangian Mechanics	#7	9/19/2018
9	Wed, 9/19/2018	Chap. 3 and 6	Lagrangian Mechanics and constraints	#8	9/24/2018
10	Fri, 9/21/2018	Chap. 3 and 6	Constants of the motion		

PHY 711 – Assignment #8

9/19/2019

Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ.)

Comments on generalized coordinates:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

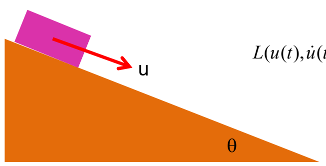
Lagrangian : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints : $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

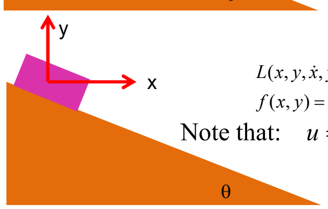
Lagrange multipliers

Modified Euler - Lagrange equations : $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Simple example:



$$L(u(t), \dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$



$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$
$$f(x, y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t), \dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$
$$f(x, y) = \sin\theta\, x + \cos\theta\, y = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$
$$\sin\theta\, \ddot{x} + \cos\theta\, \ddot{y} = 0$$
$$\Rightarrow \lambda = -mg\cos\theta$$

Force of constraint; normal to incline

$$(\cos\theta\, \ddot{x} - \sin\theta\, \ddot{y}) = g\sin\theta$$

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S=\int_{t_i}^{t_f} L(\{q_\sigma(t)\},\{\dot q_\sigma(t)\},t)\,dt$$

$$\delta S=0=\int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_\sigma}-\frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma\right)dt$$

With constraints: $f_j=f_j(\{q_\sigma(t)\},t)=0$

Variations δq_σ are no longer independent.

$$\delta f_j=0=\sum_\sigma\frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j\lambda_j\sum_\sigma\frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma$$

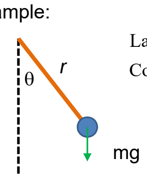
Euler-Lagrange equations with constraints:

Lagrangian: $L=L(\{q_\sigma(t)\},\{\dot q_\sigma(t)\},t)$

Constraints: $f_j=f_j(\{q_\sigma(t)\},t)=0$

Modified Euler-Lagrange equations: $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot q_\sigma}-\frac{\partial L}{\partial q_\sigma}+\sum_j\lambda_j\frac{\partial f_j}{\partial q_\sigma}=0$

Example:



Lagrangian: $L=\frac{1}{2}m\left(\dot r^2+r^2\dot\theta^2\right)+mgr\cos\theta$

Constraints: $f=r-\ell=0$

Example continued:

Lagrangian: $L=\frac{1}{2}m\left(\dot r^2+r^2\dot\theta^2\right)+mgr\cos\theta$

Constraints: $f=r-\ell=0$

$$\frac{d}{dt}m\dot r-mr\dot\theta^2-mg\cos\theta+\lambda=0$$

$$\frac{d}{dt}mr^2\dot\theta+mgr\sin\theta=0$$

$$\dot r=0=\ddot r \qquad r=\ell$$

$$\Rightarrow\ \ddot\theta=-\frac{g}{\ell}\sin\theta$$

$$\Rightarrow\ \lambda=m\ell\dot\theta^2+mg\cos\theta$$

Another example:

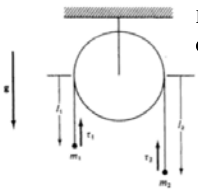


Figure 19.1 Atwood's machine.

Lagrangian: $L = \frac{1}{2}m_1\dot{\ell}_1^2 + \frac{1}{2}m_2\dot{\ell}_2^2 + m_1 g\ell_1 + m_2 g\ell_2$

Constraints: $f = \ell_1 + \ell_2 - \ell = 0$

$$\frac{d}{dt}m_1\dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt}m_2\dot{\ell}_2 - m_2 g + \lambda = 0$$

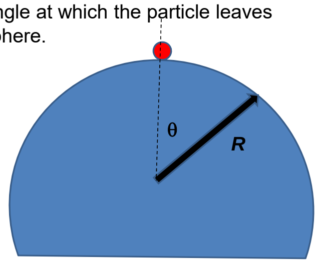
$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Another example:

A particle of mass *m* starts at rest on top of a smooth fixed hemisphere of radius *R*. Find the angle at which the particle leaves the hemisphere.



Example continued

Constraint Equation: $f(r,\theta) = r - R = 0$

Lagrangian: $L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\left(1-\cos\theta\right)$

Euler - Lagrangian equations:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \qquad mr\dot{\theta}^2 + mg\left(1-\cos\theta\right) - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Example continued

$$mr\dot{\theta}^2 + mg(1-\cos\theta) - m\ddot{r} + \lambda = 0$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Also note, from conservation of energy:

$$\frac{1}{2}mR^2\dot{\theta}^2 - mgR(1-\cos\theta) = 0$$

$$\dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)$$

Using constraint:

$$mR\dot{\theta}^2 + mg(1-\cos\theta) + \lambda = 0$$

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

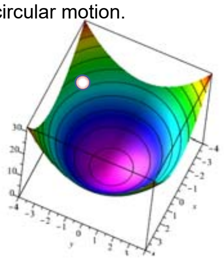
$$\ddot{\theta} = \frac{g}{R}\sin\theta \qquad \lambda = mg\cos\theta - mR\dot{\theta}^2 - mg = 3mg(\cos\theta - 1)$$

In this case λ, is not exactly the force of constraint F_C

$$F_C = mg\cos\theta - mR\dot{\theta}^2 = mg(3\cos\theta - 2) \quad \text{for } F_C \geq 0$$

$$\Rightarrow \text{for } \cos\theta = \frac{2}{3} \quad \text{or } \theta = 48.2 \text{ deg, object falls from hemisphere}$$

Consider a particle of mass *m* moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2(x\dot{x} + y\dot{y})^2\right) - mgc\left(x^2 + y^2\right)$$

$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2(x\dot{x} + y\dot{y})^2\right) - mgc\left(x^2 + y^2\right)$$

Transform to polar coordinates;

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad 0 - \frac{d}{dt}mr^2\dot{\phi} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \Rightarrow \quad \text{Let } mr^2\dot{\phi} \equiv \ell_z \quad \text{(constant)}$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Now consider the case where initially the particle is moving in a circle

at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Consider small perturbation to the motion: $r = r_0 + \delta r$

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Consider small perturbation to the motion: $r = r_0 + \delta r$

where initially the particle is moving in a circle

at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1+2c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1+2c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1+2c^2r_0^2}}t + \alpha\right)$$