PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion of Lecture 10 – Chap. 3 & 6 in F & W

Lagrangian mechanics including constraints

1. **Lagrangian representation of electromagnetic fields**
2. **Examples of Lagrangian analysis including constraints**

Physics colloquium Thursday, Sept. 17, 2020 at 4 PM

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"Drug Delivery Through the Blood-Brain Barrier, Antibody Biosensors, and Protein-Knots: Biophysics Research at an Urban-Serving Campus During the Pandemic"

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Course schedule

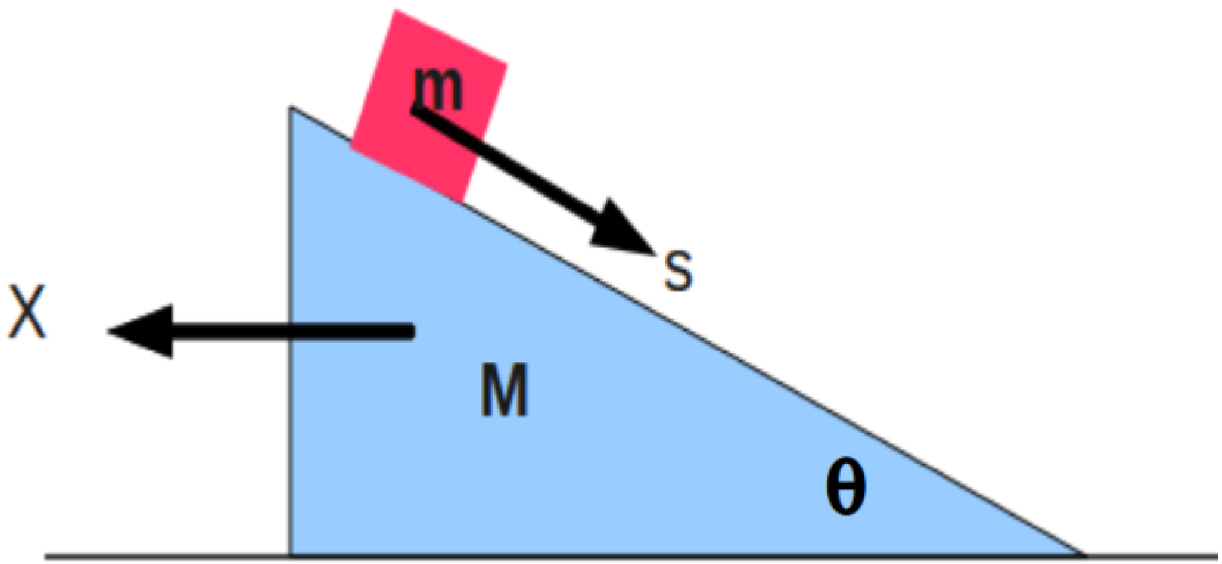
(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020

PHY 711 - Assignment #8

Sept. 16, 2020

Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ.)

Your questions –
From Nick –

1. How do you solve the Lagrangian for Euler angles presented in Lecture 9?
2. Would like some details on the Lagrangian for electromagnetic interactions.

From Tim –

1. When adding a constraint to a lagrangian, is it usually associated with the geometry of the problem (ie. r-l for the pendulum)?

From Gao –

1 No matter potential depends on velocity or not, to minimize the integral S= ∫ L()dt L must satisfy the same following equation. But L must include the velocity-dependent potential for the situation when potential depends on velocity. Right? Thank you.

Are those which minimize the action: $S = \int L(\{q_\sigma\},\{\dot{q}_\sigma\},t)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Your questions –
From Jeanette –

1. I would like to go through the Lorentz force example from last lecture, it looked like it may already be included in the lecture today. I find the examples to be more helpful than the derivations.

Example from Lecture 9 representing the motion of a symmetric top with moments of inertia I_1 and I_3 and with generalized coordinates α, β, γ (Euler angles)

Another example: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2}I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Worrisome equations, but we will develop some tricks to help us solve them.

Lagrangian mechanics with Lorentz forces

Summary of results (using cartesian coordinates)

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

scalar potential → vector potential ←

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi\left(\mathbf{r}, t\right) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

where $\mathbf{E}\left(\mathbf{r}, t\right) = -\nabla\Phi\left(\mathbf{r}, t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r}, t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r}, t\right) = \nabla \times \mathbf{A}\left(\mathbf{r}, t\right)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi\left(\mathbf{r}, t\right) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

Note that if the system also has a mechanical potential, $V(\mathbf{r})$, this is added to the electromagnetic contributions. The full Lagrangian would then be

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m\left|\dot{\mathbf{r}}\right|^2 - V(\mathbf{r}) - q\Phi(\mathbf{r}, t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Note: In our discussion of D'Alembert's virtual work analysis, we concluded that

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \qquad \text{for} \qquad L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U$$

provided that $\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}=0$

Here we examine how $\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ may not be zero and can be designed to represent velocity-dependent forces.

Do you think that it is cheating to manipulate *U* in this way?

a. Yes
b. No
c. Depends on how we define *U*

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

$x-$component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

x-component: $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} + \frac{\partial U}{\partial x} = 0$

Apparently: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi\left(\mathbf{r},t\right) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$

where $\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r},t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$

Lorentz forces, continued:

$$x-\text{component of Lorentz force}: \; F_x = q\left(E_x + \tfrac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$$

$$\text{Suppose}: \quad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Consider}: \quad F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

We want to demonstrate that the supposed form of U is consistent with the general Lorentz force given above.
Evaluating the derivatives:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

$$\text{Suppose} \quad \mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c} B_0 y\right) - \frac{q}{2c} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c} B_0 x\right) + \frac{q}{2c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Example Lorentz force -- continued

Consider formulation with different Gauge : $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}(m\dot{y}) + \frac{q}{c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

Evaluation of equations:

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \tfrac{mc}{qB_0} V_0 \cos\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$y(t) = y_0 + \tfrac{mc}{qB_0} V_0 \sin\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$z(t) = z_0 + V_{0z} t$$

Comments on generalized coordinates:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrange multipliers

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

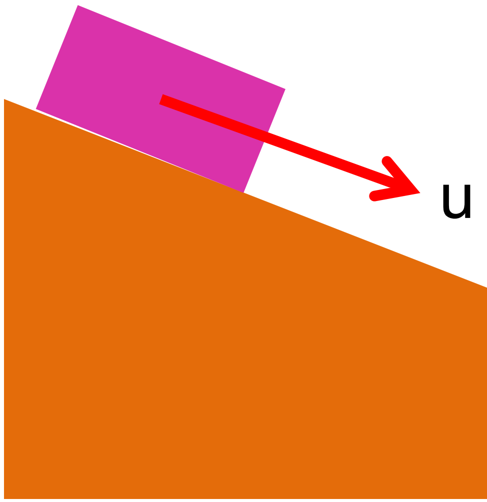
$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

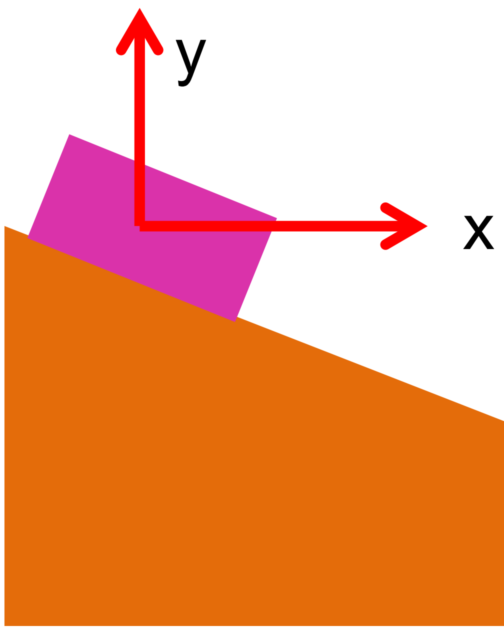
Your question -- When adding a constraint to a Lagrangian, is it usually associated with the geometry?

Comment -- That is often the case. The constraint involves relationships between the generalized coordinates and that often has a geometric interpretation. Note that in order for these equations to work, the constraint must not involve time derivatives of the coordinates. If that happens, the problem is given a terrible name – "nonholonomic" and becomes very difficult to solve.

Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$



$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0$$

$$\Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\; x + \cos\theta\; y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\;\ddot{x} + \cos\theta\;\ddot{y} = 0$$

$$\Rightarrow \lambda = -mg\cos\theta$$

$$\left(\cos\theta\;\ddot{x} - \sin\theta\;\ddot{y}\right) = g\sin\theta$$

Which method would you use to solve the problem?

Case 1

Case 2

Force of constraint; normal to incline

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

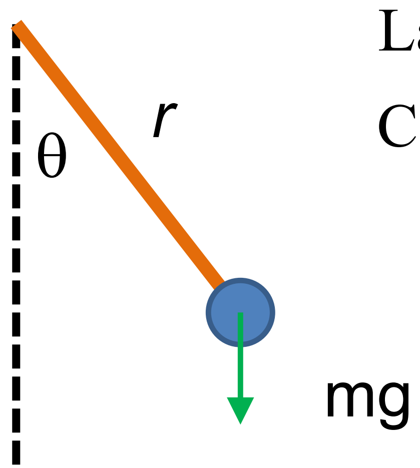
Euler-Lagrange equations with constraints:

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Example:



$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

Example continued:

Lagrangian: $L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$

Constraints: $f = r - \ell = 0$

$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad\qquad r = \ell$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \quad \lambda = m\ell\,\dot{\theta}^2 + mg\cos\theta$$

Another example:

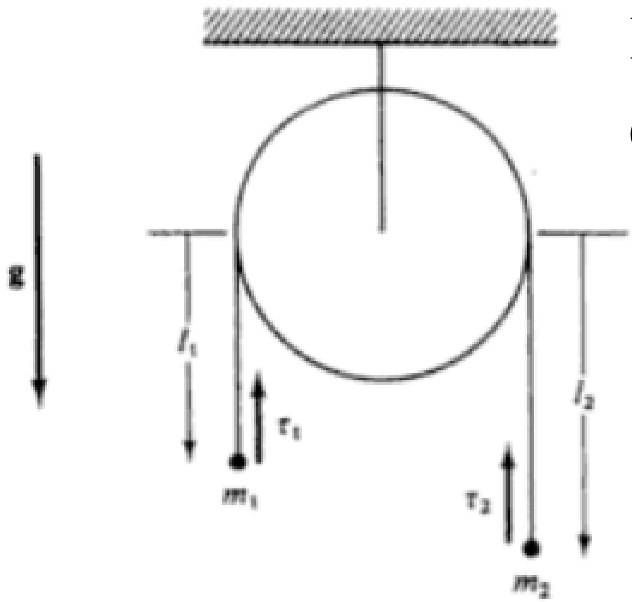


Figure 19.1 Atwood's machine.

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m_1 \dot{\ell}_1^2 + \tfrac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$$

$$\text{Constraints}: \quad f = \ell_1 + \ell_2 - \ell = 0$$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2 m_1 m_2}{m_1 + m_2} g$$

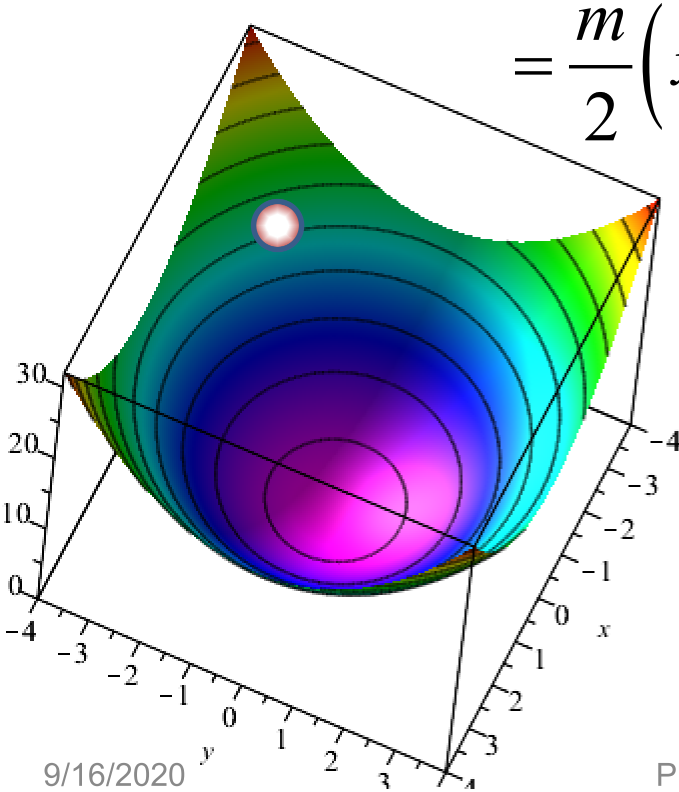
$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Consider a particle of mass *m* moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$L(x, y, \dot{x}, \dot{y})$$

$$= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$



$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

Transform to polar coordinates;

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad 0 - \frac{d}{dt}mr^2\dot{\phi} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \Rightarrow \qquad \text{Let } mr^2\dot{\phi} \equiv \ell_z \quad \text{(constant)}$$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

$$\frac{\ell_z{}^2}{mr^3} + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Now consider the case where initially the particle is moving in a circle at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Consider small perturbation to the motion: $r = r_0 + \delta r$

Some details --

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

For: $r = r_0 + \delta r$ where $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$

To linear order: $\frac{\ell_z^{\,2}}{mr^3} \approx 2mgcr_0 - 6mgc\delta r$

$$-2mgcr \approx -2mgcr_0 - 2mgc\delta r$$

$$4m\dot{r}^2c^2r \approx 0$$

$$-\frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) \approx m\delta\ddot{r}\left(1+4c^2r_0^2\right)$$

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Consider small perturbation to the motion: $r = r_0 + \delta r$

where initially the particle is moving in a circle

at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0.$

Keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1+4c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1+4c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1+4c^2r_0^2}}t + \alpha\right)$$