PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 10 – Chap. 3 & 6 in F & W

Lagrangian mechanics including constraints

1. **Lagrangian representation of electromagnetic fields**
2. **Examples of Lagrangian analysis including constraints**

In this lecture we will review Lagrangian formulations of mechanics including the effects of electromagnetic fields and also formulations with constraints.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

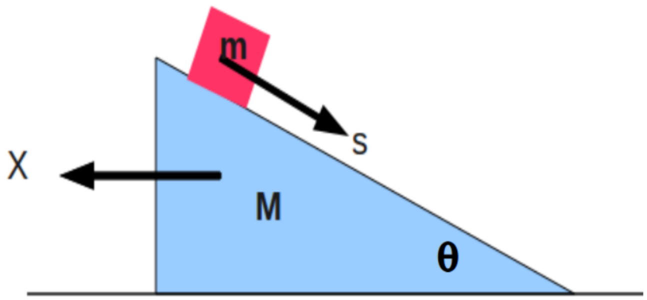
	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Constants of the motion	#8	9/21/2020

Updated schedule.

PHY 711 - Assignment #8

Sept. 16, 2020

Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ.)

Homework for Monday. Note that this problem does not include electromagnetic fields and does not need to impose constraints. It is necessary to consider the coupled motions of the two masses.

Physics colloquium Thursday, Sept. 17, 2020 at 4 PM

John Finke, PhD

Associate Professor
Sciences and Mathematics, division of
School of Interdisciplinary Arts and Sciences
University of Washington, Tacoma

"Drug Delivery Through the Blood-Brain Barrier, Antibody Biosensors, and Protein-Knots: Biophysics Research at an Urban-Serving Campus During the Pandemic"

Our colloquium speaker for Thursday's session is speaking on a very timely topic. Please let me or Kittye know if you do not receive the zoom link.

Lagrangian mechanics with Lorentz forces

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Here is a summary of the equations we “justified” at the end of the last lecture.

Note: In our discussion of D'Alembert's virtual work analysis, we concluded that

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv T - U$$

provided that $\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$=0

Here we examine how $\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ may not be zero and can represent velocity-dependent forces.

Here we consider how the equations of Lagrangian equations may be used to represent a velocity dependent force.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

$x-$component: $F_x = q\left(E_x + \frac{1}{c}\left(\mathbf{v}\times\mathbf{B}\right)_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

x-component: $$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} + \frac{\partial U}{\partial x} = 0$$

Apparently: $$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

Answer: $$U = q\Phi\left(\mathbf{r},t\right) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

where $$\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r},t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$$

Here are given the results without proof. In the following slides we will show that this form of the Lagrangian is equivalent to Newton’s laws.

Lorentz forces, continued:

$x-$component of Lorentz force: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

Suppose: $U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Consider: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Detailed steps in the analysis.

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Putting all of the terms together. In the last line we express the scalar and vector potentials in terms of the electric and magnetic field components.

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

$$L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Summary of what the previous analysis showed.

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Suppose} \quad \mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Here we consider a particular example of a particle moving in the presence of a magnetic field along the z direction.

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Coupled equations of motion.

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Finally the equations show that the particle is moving in a circular trajectory in the x-y plane.

Example Lorentz force -- continued

Consider formulation with different Gauge : $\mathbf{A}(\mathbf{r}) = -B_0 y\hat{\mathbf{x}}$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c}B_0 y\right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}(m\dot{y}) + \frac{q}{c}B_0\dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt}m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

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Now consider the same magnetic field but use a different vector potential. Do you think that the result should be the same as the previous case?

Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \tfrac{mc}{qB_0} V_0 \cos\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$y(t) = y_0 + \tfrac{mc}{qB_0} V_0 \sin\left(\tfrac{qB_0}{mc} t + \phi\right)$$

$$z(t) = z_0 + V_{0z} t$$

Here we see that the results are equivalent. Evaluating the equations for particular initial conditions, we find explicit functions for the trajectories.

Comments on generalized coordinates:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrangian: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

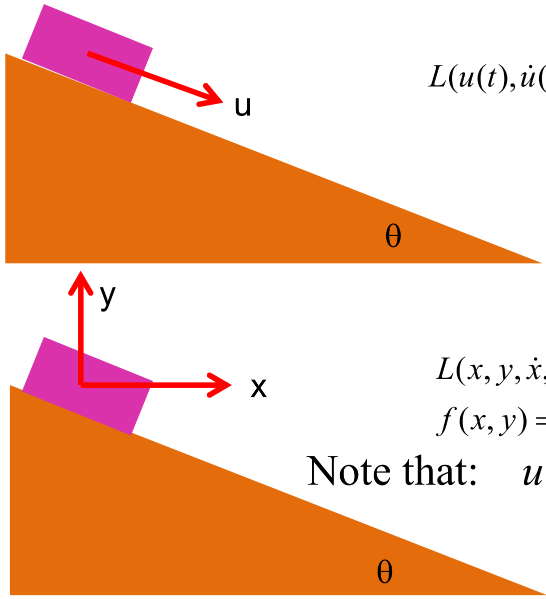
Constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Lagrange multipliers

Modified Euler-Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Shifting topics, we now consider examples where the generalized coordinates are related by some constraints.

Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Here is a simple example of an inclined plane. If we were so silly as to treat the x and y motions separately, we would have use a constraint equation as shown.

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\; x + \cos\theta\; y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\;\ddot{x} + \cos\theta\;\ddot{y} = 0$$

$$\Rightarrow \lambda = -mg\cos\theta$$

Force of constraint; normal to incline

$$\left(\cos\theta\;\ddot{x} - \sin\theta\;\ddot{y}\right) = g\sin\theta$$

In this case we see that the constraint is related to the normal force which can be considered as a force of constraint.

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

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Here we justify the use of Lagrange multipliers in a similar way that we used them when discussing the calculus of variation.

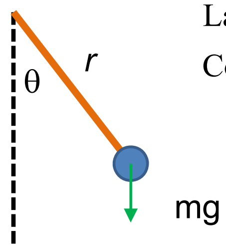
Euler-Lagrange equations with constraints:

Lagrangian: $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

Modified Euler - Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Example:



Lagrangian: $L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$

Constraints: $f = r - \ell = 0$

Another example of constrained motion.

Example continued:

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad r = \ell$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \quad \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$

Continued analysis of pendulum motion

Another example:

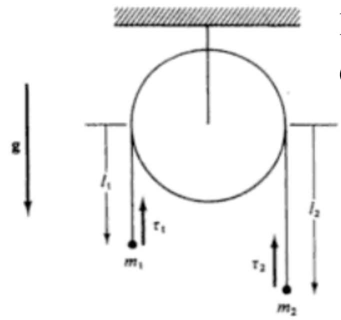


Figure 19.1 Atwood's machine.

Lagrangian: $L = \frac{1}{2} m_1 \dot{\ell}_1^2 + \frac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$

Constraints: $f = \ell_1 + \ell_2 - \ell = 0$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

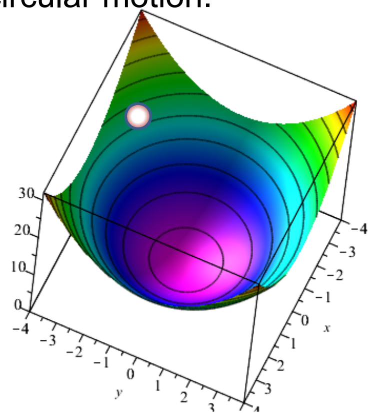
$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Example of Atwood's machine with two masses and a pulley.

Consider a particle of mass *m* moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

Another example.

$$L(x,y,\dot{x},\dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 4c^2\left(x\dot{x} + y\dot{y}\right)^2\right) - mgc\left(x^2 + y^2\right)$$

Transform to polar coordinates;

$x = r\cos\phi \qquad y = r\sin\phi$

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad 0 - \frac{d}{dt}mr^2\dot{\phi} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \Rightarrow \quad \text{Let } mr^2\dot{\phi} \equiv \ell_z \quad \text{(constant)}$$

Example continued.

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

$$\frac{\ell_z^{\ 2}}{mr^3} + 4m\dot{r}^2c^2r - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Now consider the case where initially the particle is moving in a circle at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$.

Consider small perturbation to the motion: $r = r_0 + \delta r$

Continued

Some details --

$$\frac{\ell_z^{\;2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

For: $r = r_0 + \delta r$ where $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0$

To linear order: $\frac{\ell_z^{\;2}}{mr^3} \approx 2mgcr_0 - 6mgc\delta r$

$$-2mgcr \approx -2mgcr_0 - 2mgc\delta r$$

$$4m\dot{r}^2c^2r \approx 0$$

$$-\frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) \approx m\delta\ddot{r}\left(1+4c^2r_0^2\right)$$

Continued

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr + 4m\dot{r}^2c^2r - \frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right) = 0$$

Consider small perturbation to the motion: $r = r_0 + \delta r$

where initially the particle is moving in a circle

at height z_0 and $\ell_z = mz_0\sqrt{\frac{2g}{c}} \equiv mr_0^2\sqrt{2gc}$ with $\dot{r}_0 = 0.$

Keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1+4c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1+4c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1+4c^2r_0^2}}t + \alpha\right)$$

Approximate solution.