

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF online or (occasionally)
in Olin 103
Plan for Lecture 11-- Chap. 3 & 6 (F & W)**

Details and extensions of Lagrangian mechanics

- 1. Constants of the motion**
- 2. Conserved quantities**
- 3. Legendre transformations**

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In this lecture we will continue to explore the mathematical and physical properties of the Lagrangian formalism.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion		

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Updated schedule. Home work from Lecture 10 due Monday.

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$$

Review of the Euler-Lagrange equations with a focus on finding constants of the motion.

Examples of constants of the motion:

Example 1: one - dimensional potential :

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt}m\dot{x} = 0 \quad \Rightarrow m\dot{x} \equiv p_x \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{y} = 0 \quad \Rightarrow m\dot{y} \equiv p_y \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{z} = -\frac{\partial V}{\partial z}$$

Simple example.

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt}mr^2\dot{\phi} = 0 \quad \Rightarrow mr^2\dot{\phi} \equiv p_{\phi} \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}mr\dot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_{\phi}^2}{mr^3} - \frac{\partial V}{\partial r}$$

Another example using different coordinates.

Recall alternative form of Euler-Lagrange equations:

Starting from :

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$\begin{aligned} \text{Also note that : } \frac{dL}{dt} &= \sum_\sigma \frac{\partial L}{\partial q_\sigma} \dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt} \left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) + \frac{\partial L}{\partial t} \\ &\Rightarrow \frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t} \end{aligned}$$

Reviewing “alternative” form of Euler-Lagrange equations.

Additional constant of the motion:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 1: one - dimensional potential :

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(z) \right) = -E \quad (\text{constant})$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \dot{z}^2 + V(z)$$

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New constant of the motion found from the Lagrangian analysis.

Additional constant of the motion -- continued:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) - m\dot{r}^2 - mr^2 \dot{\phi}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \right) = -E \quad (\text{constant})$$

For this case, we also have $mr^2 \dot{\phi} \equiv p_{\phi}$

$$\Rightarrow E = \frac{p_{\phi}^2}{2mr^2} + \frac{1}{2} m \dot{r}^2 + V(r)$$

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Another example.

Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{x}y + \dot{y}x)$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad m\dot{z} = p_z \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$\begin{aligned} &= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{x}y + \dot{y}x) \\ &\quad - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{2c}B_0(-\dot{x}y + \dot{y}x) \end{aligned}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{p_z^2}{2m}$$

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Non-trivial example with constant magnetic field.

Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad m\dot{z} = p_z \quad (\text{constant})$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad m\dot{x} = p_x \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}B_0\dot{x}y$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

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Another example with a constant magnetic field.

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Noting that the Lagrangian analysis generally results in second order differential equations. We now explore the possibility of alternative formulations.

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x, y) \Leftrightarrow x(y, z) ???$$

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\text{But: } \left(\frac{\partial x}{\partial y} \right)_z = - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$$

Digression on relationships between alternative coordinate formulations.

Simple change of variables -- continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y} \right)_z = - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y} \quad \Rightarrow \left(\frac{\partial x}{\partial z} \right)_y = \frac{1}{(\partial z / \partial x)_y}$$

Relationships for changing variables of general functions

Simple change of variables -- continued:

Example:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$z(x, y) = e^{x^2+y}$$

$$x(y, z) = (\ln z - y)^{1/2} \quad x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\left(\frac{\partial x}{\partial y} \right)_z \stackrel{?}{=} - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y} \quad \left(\frac{\partial x}{\partial z} \right)_y \stackrel{?}{=} \frac{1}{(\partial z / \partial x)_y}$$

$$-\frac{1}{2(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \quad \frac{1}{2z(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

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Some examples.

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$\text{Let } u \equiv \left(\frac{\partial z}{\partial x} \right)_y \quad \text{and} \quad v \equiv \left(\frac{\partial z}{\partial y} \right)_x$$

Define new function

$$w(u, y) \Rightarrow dw = \left(\frac{\partial w}{\partial u} \right)_y du + \left(\frac{\partial w}{\partial y} \right)_u dy$$

$$\text{For } w = z - ux, \quad dw = dz - udx - xdu = \cancel{u}dx + vdy - \cancel{u}dx - xdu$$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u} \right)_y = -x \quad \left(\frac{\partial w}{\partial y} \right)_u = \left(\frac{\partial z}{\partial y} \right)_x = v$$

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Relationships between the old and new variables.

For thermodynamic functions:

Internal energy: $U = U(S, V)$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S} \right)_V dS + \left(\frac{\partial U}{\partial V} \right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S} \right)_V \quad P = - \left(\frac{\partial U}{\partial V} \right)_S$$

Enthalpy: $H = H(S, P) = U + PV$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S} \right)_P dS + \left(\frac{\partial H}{\partial P} \right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S} \right)_P \quad V = \left(\frac{\partial H}{\partial P} \right)_S$$

Examples from thermodynamic functions

Name	Potential	Differential Form
Internal energy	$E(S, V, N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E, V, N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S, P, N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T, V, N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T, P, N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T, V, \mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

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Common thermodynamic energy functions.

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \text{where } p_{\sigma} = \frac{\partial L}{\partial \dot{q}_{\sigma}}$$

$$dH = \sum_{\sigma} \left(\dot{q}_{\sigma} dp_{\sigma} + p_{\sigma} d\dot{q}_{\sigma} - \frac{\partial L}{\partial q_{\sigma}} dq_{\sigma} - \frac{\partial L}{\partial \dot{q}_{\sigma}} d\dot{q}_{\sigma} \right) - \frac{\partial L}{\partial t} dt$$

Applying the ideas of variable change to the Lagrangian formulation.

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \text{where} \quad p_{\sigma} = \frac{\partial L}{\partial \dot{q}_{\sigma}}$$

$$\begin{aligned} dH &= \sum_{\sigma} \left(\dot{q}_{\sigma} dp_{\sigma} + p_{\sigma} d\dot{q}_{\sigma} - \frac{\partial L}{\partial q_{\sigma}} dq_{\sigma} - \frac{\partial L}{\partial \dot{q}_{\sigma}} d\dot{q}_{\sigma} \right) - \frac{\partial L}{\partial t} dt \\ &= \sum_{\sigma} \left(\frac{\partial H}{\partial q_{\sigma}} dq_{\sigma} + \frac{\partial H}{\partial p_{\sigma}} dp_{\sigma} \right) + \frac{\partial H}{\partial t} dt \\ \Rightarrow \dot{q}_{\sigma} &= \frac{\partial H}{\partial p_{\sigma}} \quad \frac{\partial L}{\partial q_{\sigma}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} \equiv \dot{p}_{\sigma} = -\frac{\partial H}{\partial q_{\sigma}} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \end{aligned}$$

Transforming from the Lagrangian to Hamiltonian formulation of mechanics. We will continue this discussion on Monday.