PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasional) in Olin 103

Discusion for Lecture 12 – Chap. 3&6 (F&W)

1. Constructing the Hamiltonian
2. Hamilton's canonical equation
3. Examples

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Wednesday (this week only?)
Derek – 12 PM Friday

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020

PHY 711 – Assignment #9

September 21, 2020

1. Consider a Lagrangian describing one dimensional motion of a particle of mass m in a mechanical potential $V(x)$ with an addition time dependent function $s(t)$ and extra constants Q and K having the form

$$L(x, \dot{x}, s, \dot{s}) = \frac{1}{2}m\dot{x}^2 - V(x) + Q\dot{s}^2 - K\ln(s).$$

   (a) Find the constants of motion for this system.

   (b) Find the corresponding Hamilitonian if canonical form $H(x, p_x, s, p_s)$.

Your questions –

From Jeanette –

1. Is there a physical meaning to a constant H?

From Nick –

1. Can you also explain tomorrow why the term vanishes in the equation below?

Detail :

$$\int_{t_i}^{t_f}\left(\sum_{\sigma}(\delta\dot{q}_\sigma p_\sigma)\right)dt = \int_{t_i}^{t_f}\left(\sum_{\sigma}\left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma\right)\right)dt = \overbrace{\sum_{\sigma}\delta q_\sigma p_\sigma\Big|_{t_i}^{t_f}}^{0} - \int_{t_i}^{t_f}\left(\sum_{\sigma}(\delta q_\sigma \dot{p}_\sigma)\right)dt$$

From Tim –

1. How can we make the statement H = sum d/dt(q)*p - L? Where is this equation coming from?

From Gao –

1. Where can we use the Liouville theorem, the density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion?

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Your question -- How can we make the statement H = sum d/dt(q)*p - L? Where is this equation coming from?

Comment – The reasoning for this came in the last lecture when we illustrated the notion of the Legendre transformation. Some details from Lecture 11

$$z(x,y) \quad \Rightarrow \quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy = udx + vdy$$

$$\text{for} \quad u \equiv \left(\frac{\partial z}{\partial x}\right)_y \quad \text{and} \quad v \equiv \left(\frac{\partial z}{\partial y}\right)_x$$

$$\text{Now consider} \quad w = z - ux \qquad \text{for} \quad w = w(u,y)$$

$$dw = dz - udx - xdu = udx + vdy - udx - xdu = vdy - xdu$$

$$dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy \quad \Rightarrow \quad \left(\frac{\partial w}{\partial u}\right)_y = -x \quad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

Application of the Legendre transformation for the Lagrangian and Hamiltonian

$L(q,\dot{q},t)$ and $H(q,p,t)$

suppose $H(q,p,t) = \dot{q}p - L(q,\dot{q},t)$

$$dH = \dot{q}dp + pd\dot{q} - \left(\frac{\partial L}{\partial q}\right)dq - \left(\frac{\partial L}{\partial \dot{q}}\right)d\dot{q} - \left(\frac{\partial L}{\partial t}\right)dt = \left(\frac{\partial H}{\partial q}\right)dq + \left(\frac{\partial H}{\partial p}\right)dp + \left(\frac{\partial L}{\partial t}\right)dt$$

Note that these two terms cancel if $p = \dfrac{\partial L}{\partial \dot{q}}$

The analysis on the following slides is a generalization to multiple dimensions q_σ and p_σ ….

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

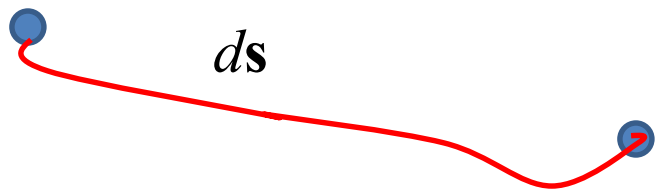
$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$\Rightarrow$ Minimization integral: $S = \int_{t_i}^{t_f} L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta \dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma} \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Canonical equations

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta \dot{q}_\sigma p_\sigma \right) \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \underbrace{\sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f}}_{0} - \int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta q_\sigma \dot{p}_\sigma \right) \right) dt$$

More comments about "details"

Detail:

$$\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta\dot{q}_\sigma p_\sigma\right)\right)dt = \int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt}-\delta q_\sigma \dot{p}_\sigma\right)\right)dt = \underbrace{\sum_\sigma \delta q_\sigma p_\sigma\Big|_{t_i}^{t_f}} - \int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta q_\sigma \dot{p}_\sigma\right)\right)dt$$

Vanishes because $\delta q(t_f)=\delta q(t_i)$ due to the premise of Hamilton's principle.

In the Hamiltonian formulation --

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Why are these equations known as the “canonical equations”?

a. Because they are beautiful.
b. The term is meant to elevate their importance to the level of the music of J. S. Bach
c. To help you remember them
d. No good reason; it is just a name

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$
2. Compute generalized momenta : $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

What happens when you miss a step in the recipe?

a. No big deal
b. Big deal – can lead to shame and humiliation (or at least wrong analysis)

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma \left(-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{ constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

Your question -- Is there a physical meaning to a constant H?

Comment -- Whenever you find a constant of the motion, it is helpful for analyzing the trajectory. In this case, H often represents the mechanical energy of the system so that constant H implies that energy is conserved.

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants: p_x, p_y, H

Equations of motion: $\dfrac{dz}{dt} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m} \qquad \dfrac{dp_z}{dt} = -\dfrac{dV}{dz}$

Here we should have done a better job of using notation to distinguish between constants and variables.

Example 2 : Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

$$p_r = m\dot{r} \qquad\qquad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)\right)$$

$$= \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants : p_ϕ, H

Equations of motion :

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad\qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \tfrac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2 \right) + \tfrac{1}{2} I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)$$

$$H = \tfrac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2 \right) + \tfrac{1}{2} I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)^2 + Mgh \cos \beta$$

$$H = \frac{\left(p_\alpha - p_\gamma \cos \beta \right)^2}{2 I_1 \sin^2 \beta} + \frac{p_\beta^2}{2 I_1} + \frac{p_\gamma^2}{2 I_3} + Mgh \cos \beta$$

Constants : p_α, p_γ, H

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

$$\text{Constants}: \qquad p_z, H$$

Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

$$\text{Constants}: \qquad p_z, H$$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right)$$

Canonical equations of motion for constant magnetic field -- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c} B_0 x \right) = \frac{qB_0}{2c}\frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c} B_0 y \right) = -\frac{qB_0}{2c}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

Are these results equivalent to the results of the Lagrangian analysis?

a. Yes
b. No

General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

Lagrangian: $$L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Hamiltonian: $$\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Some details: $L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Hamiltonian: $\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\left(m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)\cdot\dot{\mathbf{r}}-\left(\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)$$

$$=\frac{1}{2}m\dot{\mathbf{r}}^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Canonical form

Other properties of Hamiltonian formalism – Poisson brackets:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \; q_\sigma \; \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \; p_\sigma \; \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function : $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

So that : $\dfrac{dF}{dt} = [F,H]_{PB} + \dfrac{\partial F}{\partial t}$

Poisson brackets -- continued:

$$[F,G]_{PB} \equiv \sum_{\sigma}\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma}\right) = -[G,F]_{PB}$$

Examples:

$$[x,x]_{PB} = 0 \qquad [x,p_x]_{PB} = 1 \qquad [x,p_y]_{PB} = 0$$

$$[L_x,L_y]_{PB} = L_z$$

Liouville theorem

Let $D \equiv$ density of particles in phase space:

$$\frac{dD}{dt} = [D,H]_{PB} + \frac{\partial D}{\partial t} = 0$$

Your question -- Where can we use the Liouville theorem, the density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion?

Comment – This idea is particularly important in statistical mechanics where the challenge is to describe macroscopic phenomena based on 10^{23} particles in terms of our knowledge of individual particles and their interactions.

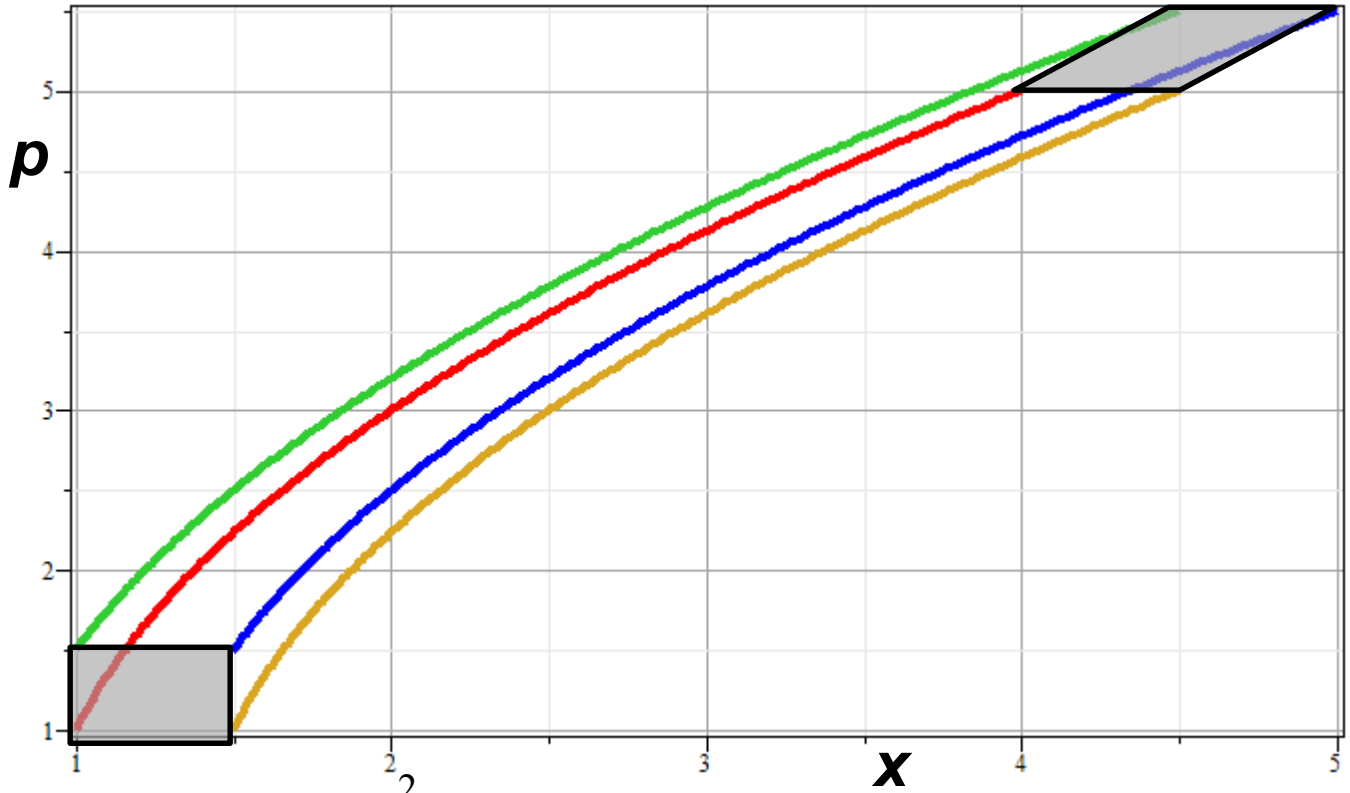
Phase space

Phase space is defined at the set of all coordinates and momenta of a system:

$$\left(\{q_\sigma(t)\},\{p_\sigma(t)\}\right)$$

For a d dimensional system with N particles, the phase space corresponds to $2dN$ degrees of freedom.

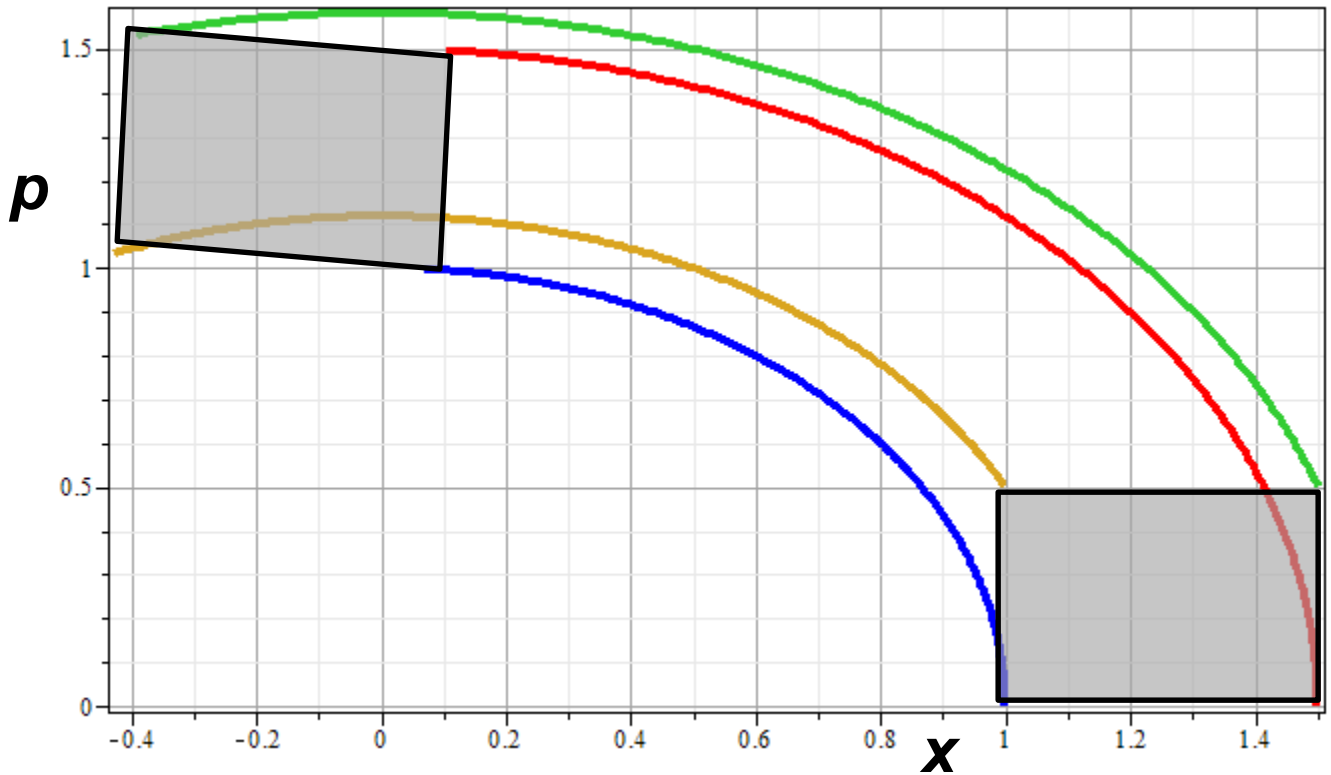
Phase space diagram for one-dimensional motion due to constant force



$$H(x,p) = \frac{p^2}{2m} - F_0 x \qquad \dot{p} = F_0 \qquad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i} + F_0 t \qquad x_i(t) = x_{0i} + \frac{p_{0i}}{m} t + \frac{1}{2} F_0 t^2$$

Phase space diagram for one-dimensional motion due to spring force



$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad \dot{p} = -m\omega^2 x \qquad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i}\cos(\omega t + \theta_{0i}) \qquad x_i(t) = \frac{p_{0i}}{m\omega}\sin(\omega t + \theta_{0i})$$

Liouville's Theorem (1838)

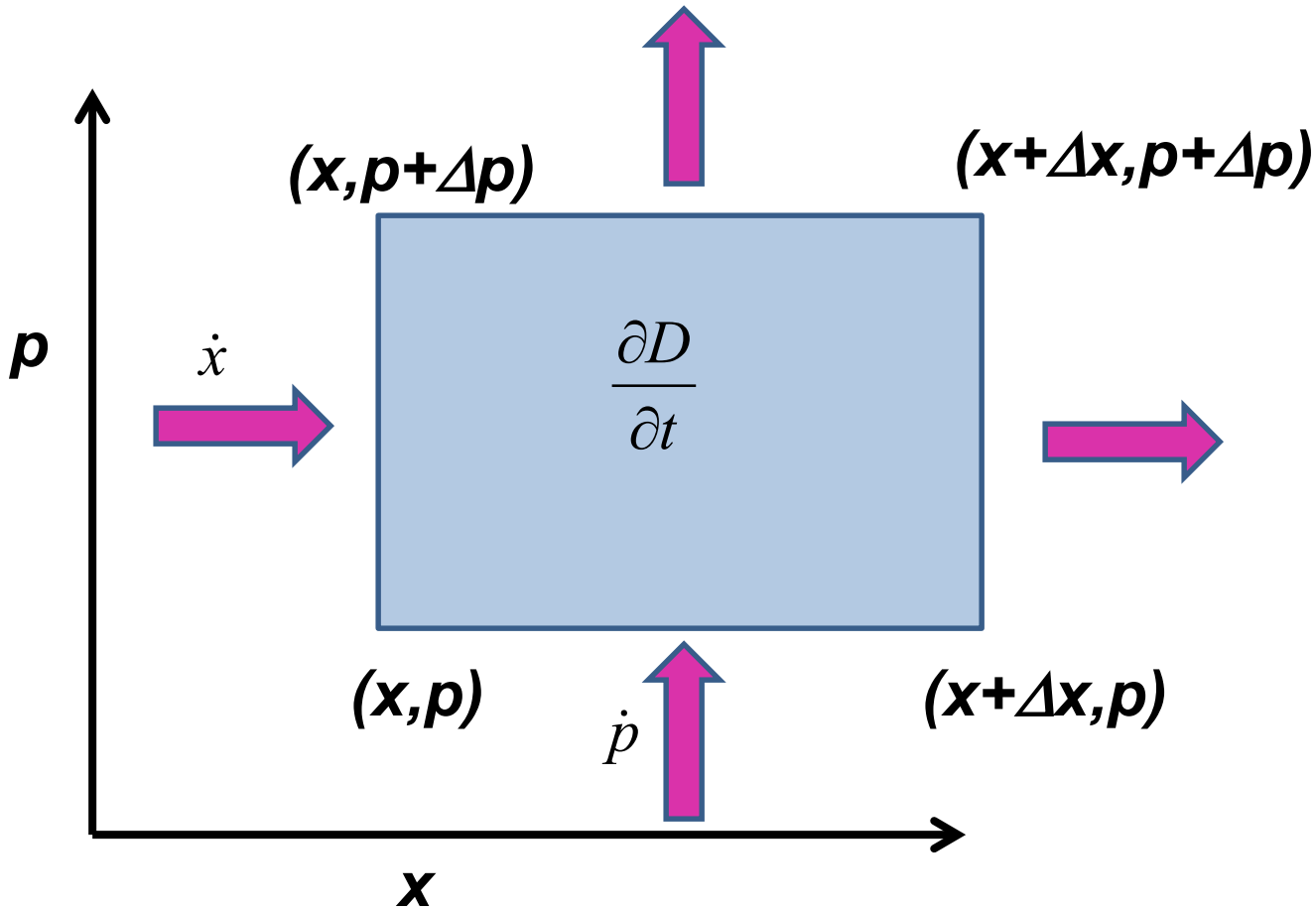
The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

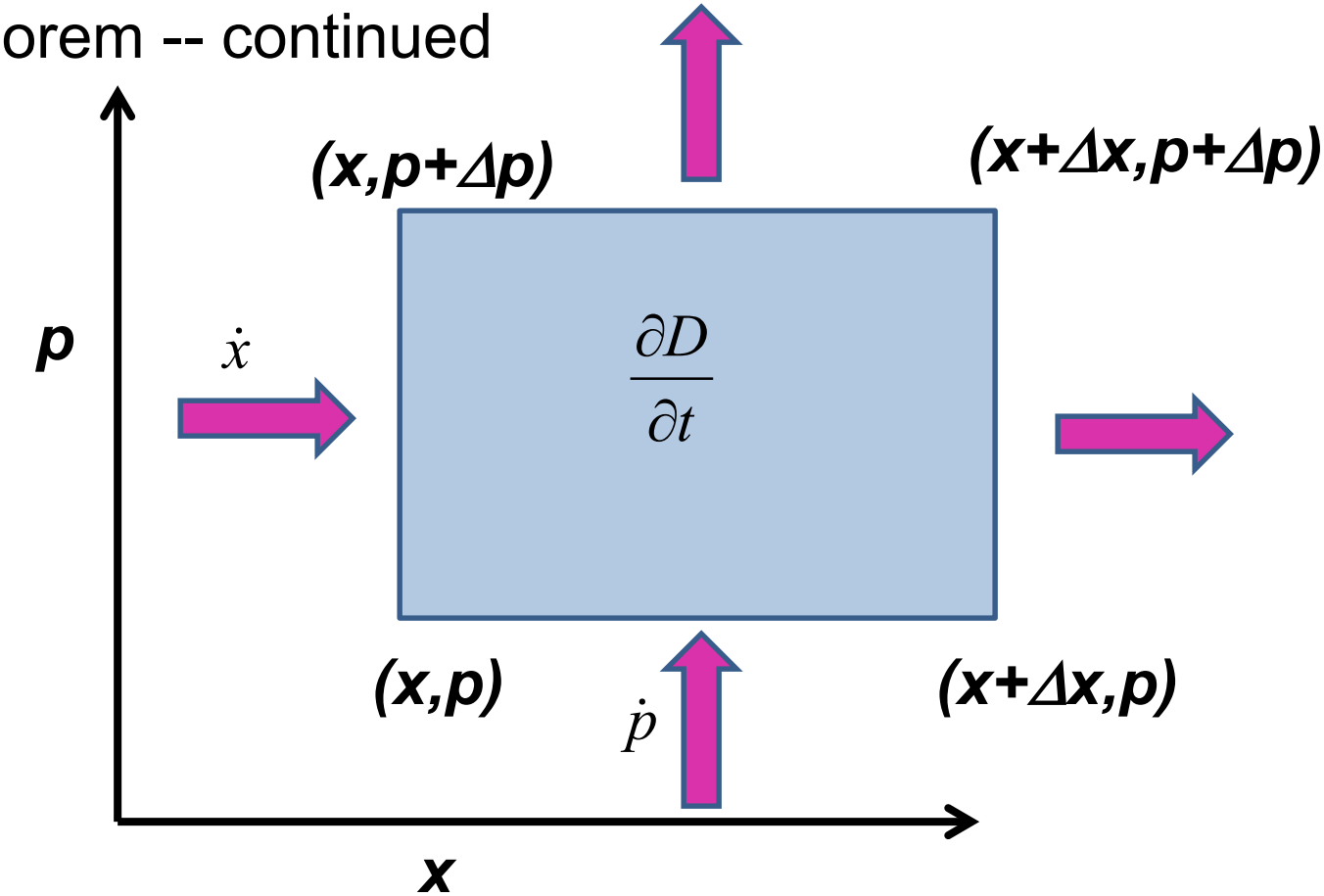
$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\frac{dD}{dt} = 0$

Liouville's theorem



Liouville's theorem -- continued

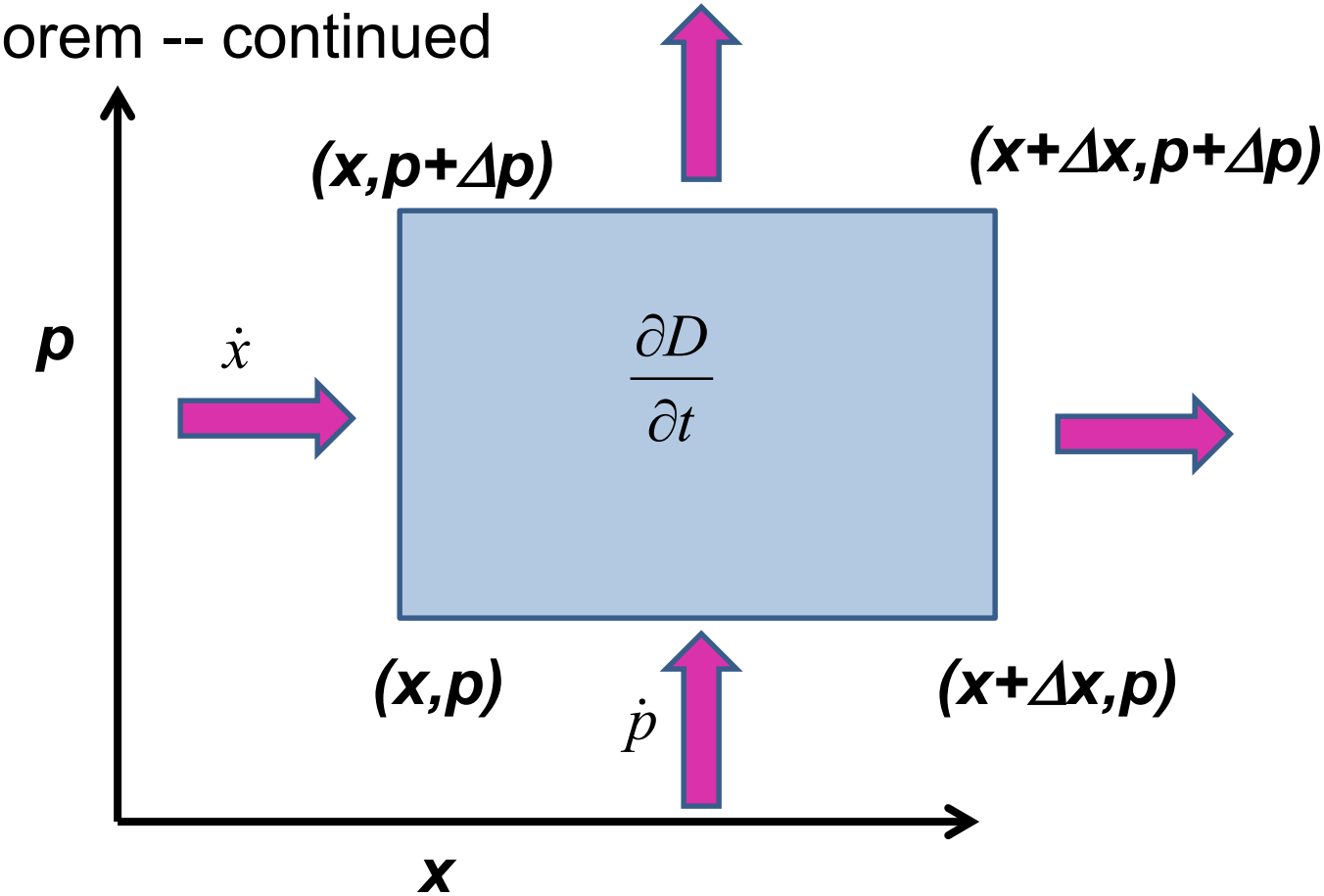


$$\frac{\partial D}{\partial t} \Rightarrow \text{time rate of change of particles within volume}$$

$$= \text{ time rate of particle entering minus particles leaving}$$

$$= -\frac{\partial D}{\partial x}\dot{x} - \frac{\partial D}{\partial p}\dot{p}$$

Liouville's theorem -- continued



$$\frac{\partial D}{\partial t} = -\frac{\partial D}{\partial x}\dot{x} - \frac{\partial D}{\partial p}\dot{p}$$

$$\frac{\partial D}{\partial t} + \frac{\partial D}{\partial x}\dot{x} + \frac{\partial D}{\partial p}\dot{p} = 0 = \frac{dD}{dt}$$

Review:

Liouville's theorem:

Imagine a collection of particles obeying the Canonical equations of motion in phase space.

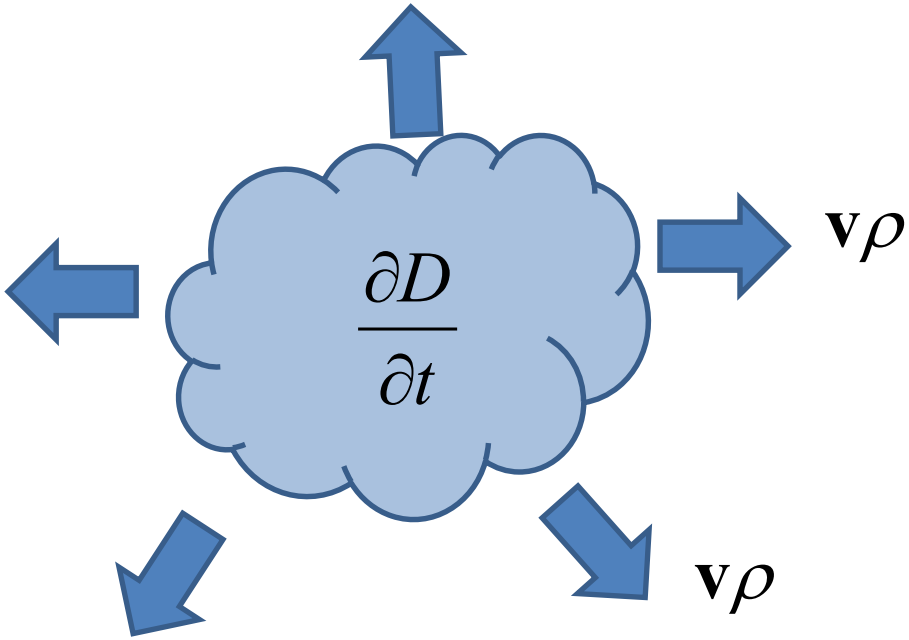
Let D denote the "distribution" of particles in phase space:

$$D = D\left(\{q_1 \cdots q_{3N}\}, \{p_1 \cdots p_{3N}\}, t\right)$$

Liouville's theorm shows that:

$$\frac{dD}{dt} = 0 \qquad \Rightarrow D \text{ is constant in time}$$

Proof of Liouville's theorem:



Continuity equation :

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v} D)$$

Note : in this case, the velocity is the $6N$ dimensional vector :

$$\mathbf{v} = (\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots \dot{\mathbf{p}}_N)$$

We also have a $6N$ dimensional gradient :

$$\nabla = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \ldots \nabla_{\mathbf{r}_N}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \ldots \nabla_{\mathbf{p}_N})$$

$$\frac{\partial D}{\partial t} = -\nabla \cdot \left(\mathbf{v} D\right)$$

$$= -\sum_{j=1}^{3N}\left[\frac{\partial}{\partial q_j}\left(\dot{q}_j D\right) + \frac{\partial}{\partial p_j}\left(\dot{p}_j D\right)\right]$$

$$= -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]$$

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j}\right) = 0$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right] \to 0$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right]$$

$$\Rightarrow \frac{\partial D}{\partial t} + \sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] = \frac{dD}{dt} = 0$$

$$\boxed{\frac{dD}{dt} = 0}$$

Importance of Liouville's theorem to statistical mechanical analysis:

In statistical mechanics, we need to evaluate the probability of various configurations of particles. The fact that the density of particles in phase space is constant in time, implies that each point in phase space is equally probable and that the time average of the evolution of a system can be determined by an average of the system over phase space volume.