PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 16 – Chap. 4 (F & W)

Analysis of motion near equilibrium

1. **Normal modes of vibration for simple systems**
2. **Some concepts of linear algebra**
3. **Normal modes of vibration for more complicated systems**

In this lecture we will analyze systems near equilibrium. This system represents a lot of physical systems and has a rich toolbox of mathematical formalisms.

Physics Colloquium Thursday, October 1, 2020

Online Colloquium: “Designer defects: engineering color centers in crystals as nanoscale optical sensors” — October 1, 2020 at 4 PM

Dr. Claire Allison McLellan
Wu Tsai Postdoctoral Scholar, Dionne Laboratory
Stanford University, Stanford, California
Wake Forest University Alum
Thursday, October 1, 2020 at 4:00 PM

Dr. McLellan recommends the following published papers from her group for topical information:
https://pubs.acs.org/doi/abs/10.1021/acscentsci.9b00300
https://pubs.acs.org/doi/abs/10.1021/acs.nanolett.5b05304

A review paper on this topic may also be of interest:
https://www.nature.com/articles/s41586-020-2048-8

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Thursday’s colloquium -- WFU alum

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020

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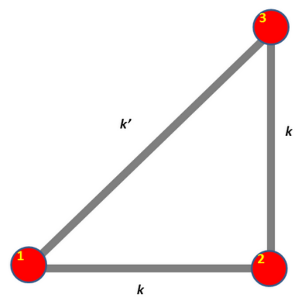
We are starting the material covered in Chap. 4. The assigned homework will be covered in Friday's lecture and due on Monday.

PHY 711 -- Assignment #12

Sept. 30, 2020

Finish reading Chapter 4 in **Fetter & Walecka**.



1. Consider the system of 3 masses (m_1=m_2=m_3=m) shown attached by elastic forces in the right triangular configuration (with angles 45, 90, 45 deg) shown above with spring constants k and k'. Find the normal modes of small oscillations for this system. For numerical evaluation, you may assume that k=k'.

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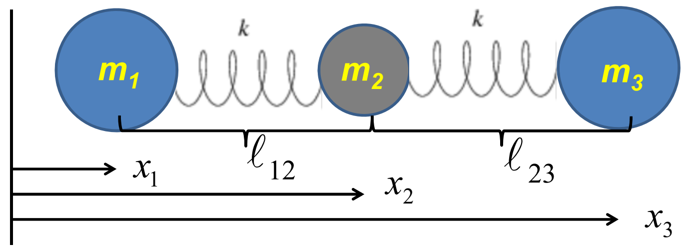
Extension of ideas discussed today to 2 dimensions.

The following slides quickly review what we covered in Lecture 15 --

Non-trivial example of coupled oscillator in the form of a linear molecule



$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$
$$- \frac{1}{2} k \left(x_2 - x_1 - \ell_{12} \right)^2 - \frac{1}{2} k \left(x_3 - x_2 - \ell_{23} \right)^2$$

Example with 3 masses connected with springs moving in one dimension

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$$
$$-\frac{1}{2}k\left(x_2 - x_1 - \ell_{12}\right)^2 - \frac{1}{2}k\left(x_3 - x_2 - \ell_{23}\right)^2$$

Let: $x_1 \to x_1 - x_1^0 \quad x_2 \to x_2 - x_1^0 - \ell_{12} \quad x_3 \to x_3 - x_1^0 - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2 - \frac{1}{2}k\left(x_3 - x_2\right)^2$$

Coupled equations of motion using simplified variables:

$$m_1\ddot{x}_1 = k\left(x_2 - x_1\right)$$
$$m_2\ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) = k\left(x_1 - 2x_2 + x_3\right)$$
$$m_3\ddot{x}_3 = -k\left(x_3 - x_2\right)$$

Analyzing the equations of motion.

Coupled equations of motion :

$$m_1\ddot{x}_1 = k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3\ddot{x}_3 = -k(x_3 - x_2)$$

Mathematical methods for solving these coupled linear differential equations:

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

Coupled differential equation and tricks for solution.

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix}\begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

Resulting linear equations also written in matrix form.

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i} X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11}-\omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22}-\omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33}-\omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

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The original equations are not symmetric. With this transformation, we can make the equations take a symmetric form.

Rearranging the equation to an eigenvalue problem:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Special case for CO_2 molecule -- $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$

$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

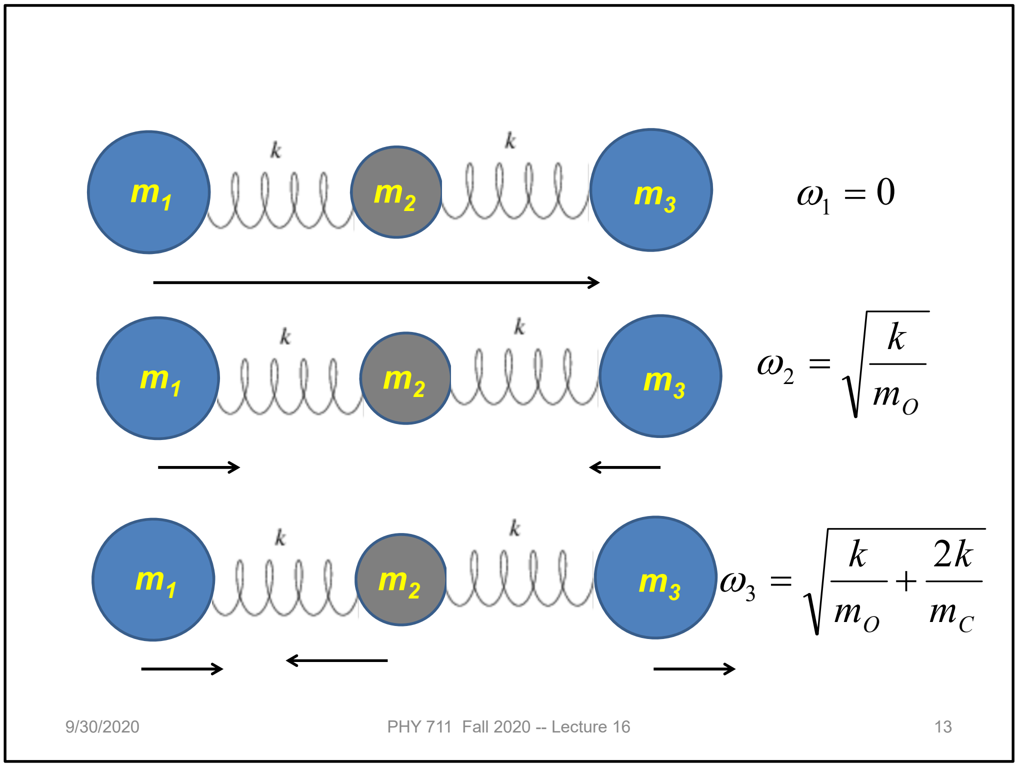
More details for our case.

Eigenvalues and eigenvectors: (with help from Maple)

$$\omega_1^2 = 0 \qquad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \qquad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \qquad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \qquad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



Visualization of the solution for our case.

Which of the following statements are false?

a. Molecules always have one zero frequency mode.
b. CO_2 really has more than 3 normal modes .
c. Some molecules have more than one zero frequency modes.
d. The normal modes of molecules are only of academic interest and cannot be measured.

Finding eigenvalues/eigenvectors by hand --

$$\mathbf{M}\mathbf{y}^\alpha = \lambda^\alpha \mathbf{y}^\alpha$$

$$\left(\mathbf{M} - \lambda^\alpha \mathbf{I}\right)\mathbf{y}^\alpha = 0$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| \equiv \det\left(\mathbf{M} - \lambda^\alpha \mathbf{I}\right) = 0 \quad \Rightarrow \text{polynomial for solutions } \lambda^\alpha$$

For each α and λ^α solve for the eigenvector coefficients $\mathbf{y}^\alpha$

Example

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A - \lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B - \lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A - \lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A + 2B)\right)$$

In general, we will want help from computational tools, but we can illustrate the concepts for 3 dimensions.

Example -- continued

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A-\lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A+2B)\right)$$

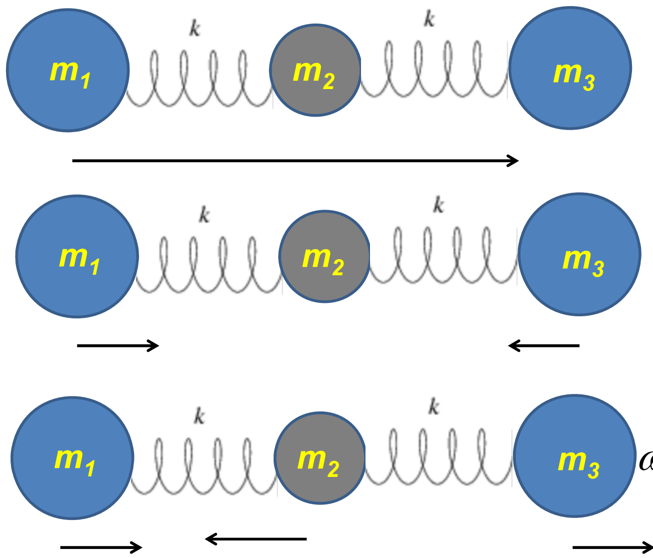
Solving for eigenvector corresponding to $\lambda^\alpha \equiv \lambda^1 = 0$

$$\begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \begin{pmatrix} y^1_{O1} \\ y^1_C \\ y^1_{O2} \end{pmatrix} = 0 \qquad \Rightarrow \frac{y^1_{O1}}{y^1_C} = \frac{y^1_{O2}}{y^1_C} = \sqrt{\frac{B}{A}}$$

Note that the normalization of the eigenvector is arbitrary.

Finding the polynomial of the eigenvalues and solving, followed by solving for the eigenvector components.

Summary of results for linear CO_2 molecule



$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k}{m_O}}$$

$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

Summary of the results for this case.

General solution :

$$x_i(t) = \Re\left(\sum_\alpha C^\alpha X_i^\alpha e^{-i\omega_\alpha t} \right)$$

For example, normal mode amplitudes

C^α can be determined from initial conditions

The general solution will depend on initial values or boundary values.

Digression:

Eigenvalue properties of matrices $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix: $\mathbf{H}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$ $\quad H_{ij} = H^*_{ji}$

Theorem for Hermitian matrices:

λ_α have real values and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

Unitary matrix: $\mathbf{U}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$ $\quad \mathbf{U}\mathbf{U}^H = \mathbf{I}$

$|\lambda_\alpha| = 1$ and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

Digression on linear algebra theory.

Digression on matrices -- continued

Eigenvalues of a matrix are "invariant" under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}\left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right) = \lambda'_\alpha \left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right)$$

This means that if a matrix is "similar" to a Hermitian matrix, it has the same distribution of eigenvalues.

Similarity transformations used to analyze our system.

Example of a similarity transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

Let $\mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}$; $\mathbf{SMS}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$

Let $\mathbf{Y} \equiv \mathbf{SX}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

Details for our case..

Note, here we have defined **S** as a transformation matrix (often called a similarity transformation matrix)

Sometimes, the similarity transformation is also unitary so that

$$\mathbf{U}^{-1} = \mathbf{U}^{H}$$

Example for 2x2 case --

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{U}^{-1} = \mathbf{U}^{H} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How can you find the transformation that diagonalizes a matrix?

Example -- $\mathbf{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \mathbf{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Another example of similarity transformation for the 2x2 case.

Example -- $\mathbf{M}=\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ $\quad \mathbf{M}'=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\mathbf{M}' = \mathbf{UMU}^H$ $\quad$ for $\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\mathbf{M}' = \begin{pmatrix} A\cos^2\theta + C\sin^2\theta + B\sin 2\theta & -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta \\ -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta & A\sin^2\theta + C\cos^2\theta - B\sin 2\theta \end{pmatrix}$$

$\Rightarrow$ choose $\theta = \tan^{-1}\left(\frac{-2B}{C-A}\right)$

$\Rightarrow \lambda_1 = A\cos^2\theta + C\sin^2\theta + B\sin 2\theta$

$\Rightarrow \lambda_2 = A\sin^2\theta + C\cos^2\theta - B\sin 2\theta$

Note that this “trick” is special for 2x2 matrices, but numerical extensions based on the trick are possible.

Only for 2x2 case.

Note that transformations using unitary matrices are often convenient and they can be easily constructed from the eigenvalues of a matrix.

Suppose you have an $N \times N$ matrix $\mathbf{M}$ and find all N eigenvalues/vectors:

$\mathbf{M}\mathbf{y}^\alpha = \lambda^\alpha \mathbf{y}^\alpha$ orthonormalized so that $\langle \mathbf{y}^\alpha | \mathbf{y}^\beta \rangle = \delta_{\alpha\beta}$

Now construct an $N \times N$ matrix $\mathbf{U}$ by listing the eigenvector columns:

$$\mathbf{U} \equiv \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^N \\ y_2^1 & y_2^2 & \cdots & y_2^N \\ \vdots & \vdots & \cdots & \vdots \\ y_N^1 & y_N^2 & \cdots & y_N^N \end{pmatrix} \qquad \mathbf{U}^{-1} \equiv \begin{pmatrix} y_1^{1*} & y_2^{1*} & \cdots & y_N^{1*} \\ y_1^{2*} & y_2^{2*} & \cdots & y_N^{N*} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{N*} & y_2^{N*} & \cdots & y_N^{N*} \end{pmatrix} \qquad \Rightarrow \text{ by construction } \mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$$

Also by construction $\mathbf{U}^{-1}\mathbf{M}\mathbf{U} = \begin{pmatrix} \lambda^1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda^N \end{pmatrix}$

Comment on unitary matrices in general.

Additional digression on matrix properties
Singular value decomposition

It is possible to factor any real matrix $\mathbf{A}$ into unitary matrices $\mathbf{V}$ and $\mathbf{U}$ together with positive diagonal matrix $\mathbf{\Sigma}$:

$$\mathbf{A} = \mathbf{U\Sigma V^H}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}$$

An unrelated digression that may be useful – singular value decomposition.

Singular value decomposition -- continued

Consider using SVD to solve a singular linear algebra problem $\mathbf{AX} = \mathbf{B}$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$$

$$\mathbf{X} = \sum_{i \text{ for } \sigma_i > \varepsilon} \mathbf{v}_i \frac{\langle \mathbf{u}_i^H \mid \mathbf{B} \rangle}{\sigma_i}$$

Details are complicated ….

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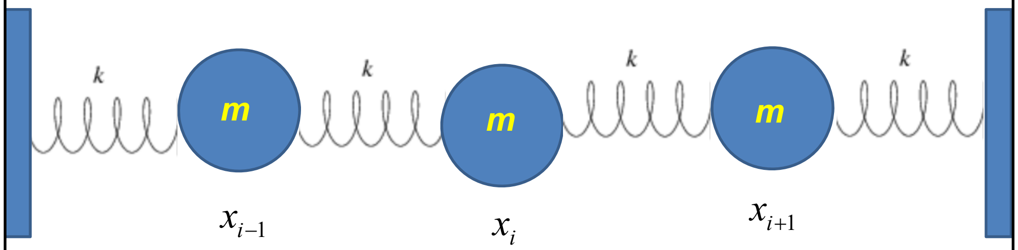
Digression continued.

Consider an extended system of masses and springs:



Note : each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} which will be treated using boundary conditions.

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Another example; this one is in your textbook.

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k\left(x_2 - 2x_1\right)$$

$$m\ddot{x}_2 = k\left(x_3 - 2x_2 + x_1\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$m\ddot{x}_N = k\left(x_{N-1} - 2x_N\right)$$

Details for N masses.

Matrix formulation --

Assume $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k}\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem –

(Why did we not have to transform the equations as we did in the previous example?)

Form of matrix equations.

```
> with(LinearAlgebra);
```

$$> M := \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix};$$

$$> Eigenvalues(M); \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2-\sqrt{3} \\ 2+\sqrt{3} \end{bmatrix}$$

Finding eigenvalues with Maple.

This example also has an algebraic solution --

From Euler - Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try:} \quad x_j(t) = Ae^{-i\omega t + iqaj}$$

$$-\omega^2 Ae^{-i\omega t + iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t + iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Is this treatment cheating?

a. Yes.
b. No cheating, but we are not done.

Analytic methods for this highly symmetric case.

From Euler-Lagrange equations -- continued:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try: $x_j(t) = Ae^{-i\omega t + iqaj}$ $\quad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

Note that: $x_j(t) = Be^{-i\omega t - iqaj}$ $\quad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

General solution:

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Setting the boundary values.

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t} \right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)} \right) = 0$$

$$\Rightarrow B = -A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t} \left(e^{iqa(N+1)} - e^{-iqa(N+1)} \right) \right) = 0$$

$$\Rightarrow \sin\left(qa(N+1) \right) = 0$$

$$\Rightarrow \; qa(N+1) = \nu\pi \quad \text{where } \nu = 0,1,2\cdots$$

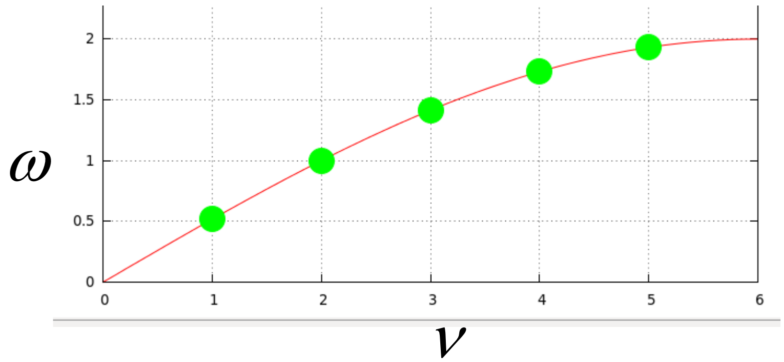
$$qa = \frac{\nu\pi}{N+1}$$

Boundary conditions continued.

Summary of results:

$$\Rightarrow \omega_\nu^2 = \frac{4k}{m}\sin^2\left(\frac{\nu\pi}{2(N+1)}\right) \qquad x_n = \Re\left(2iA\sin\left(\frac{\nu\pi n}{N+1}\right)\right)$$

$$\nu = 0,1,\ldots N \qquad n = 1,2,\ldots..N$$



Plot of the results.