PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion for Lecture 18: Chap. 7 (F&W)

Mechanical motion of a continuous string

1. **Masses coupled by springs ←→masses coupled by string**
2. **Mechanics one-dimensional continuous system**
3. **The wave equation**
4. **Comments on linear vs. non-linear equations**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
17	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations		

PHY 711 – Assignment #13

10/05/2020

Start reading Chapter 7 in **Fetter and Walecka**.

Consider a one-dimensional wave characterized by displacement $\mu(x,t)$ as a function of position x and time t is described by the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0, \tag{1}$$

where c denotes the wave speed. Find the functional form of $\mu(x,t)$ for each of these initial conditions.

1. At $t = 0$,

$$\mu(x,0) = \frac{A}{\cosh(x)} \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = 0, \tag{2}$$

   where A is a given amplitude.

2. At $t = 0$,

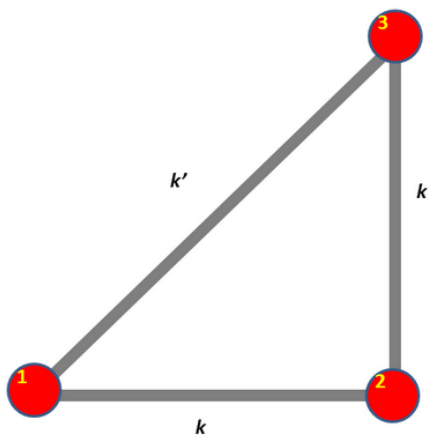
$$\mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = \frac{A \sinh(x)}{\cosh^2(x)}. \tag{3}$$

Comment about Assignment #12 due today.

PHY 711 -- Assignment #12

Sept. 30, 2020

Finish reading Chapter 4 in **Fetter & Walecka**.



Please set up the 6x6 matrix and solve for the eigenvalues using Maple, Mathematica, or Wolfram, or…??

1. Consider the system of 3 masses (m_1=m_2=m_3=m) shown attached by elastic forces in the right triangular configuration (with angles 45, 90, 45 deg) shown above with spring constants k and k'. Find the normal modes of small oscillations for this system. For numerical evaluation, you may assume that k=k'.

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 12 PM Friday

Your questions –

From Jeanette

1. On slide 6 - Is this saying that the system parameter doesn't simplify for the longitudinal case? By simplifying I mean it can be rewritten in other terms (like tension and mass/length for the transverse case). Does it only simplify in the transverse case?

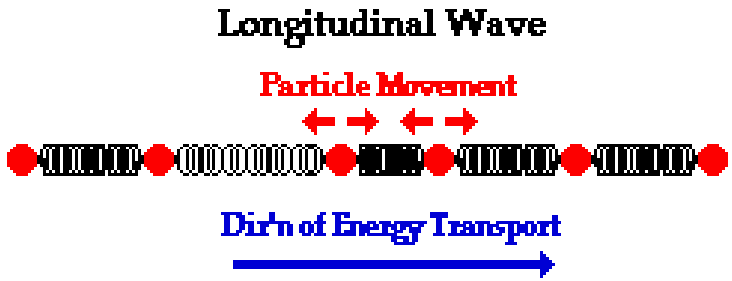
From Gao

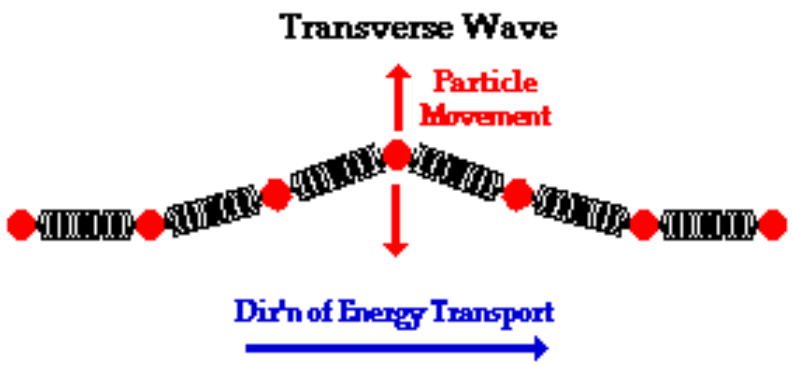
1. About lecture 18, we do not know Phi(x) and Psi(x), how can we solve U(x,0)? Or how can we get Phi(x) and Psi(x)?

Continuum limit of masses and springs --
Longitudinal versus transverse vibrations

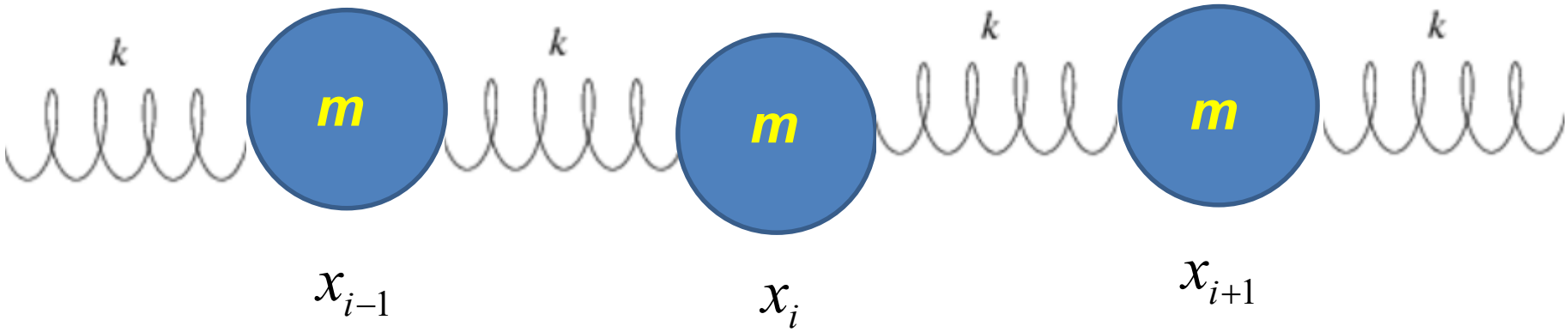
Images from web page:

http://www.physicsclassroom.com/class/waves/u10l1c.cfm





Longitudinal case: a system of masses and springs:



$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i \right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1} \right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x_i, t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

Discrete equation : $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation : $m\dfrac{\partial^2 \mu}{\partial t^2} = k\left(\Delta x\right)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2

For transverse oscillations on a string with tension τ and mass/length σ:

$$\left(\frac{k\Delta x}{m/\Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

More details

Longitudinal case

Consider Taylor's series

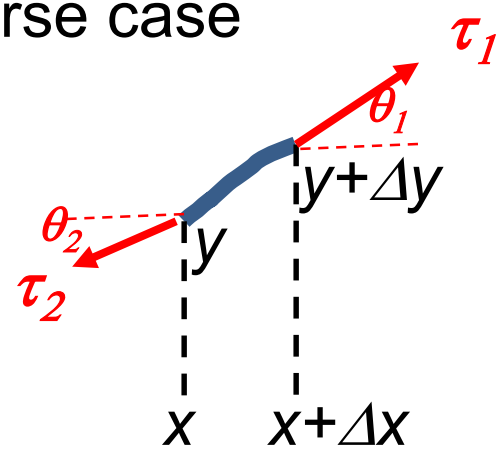
$$x(t+\Delta t) = x(t) + \Delta t \left.\frac{dx}{dt}\right|_t + \frac{1}{2}(\Delta t)^2 \left.\frac{d^2x}{dt^2}\right|_t + \frac{1}{6}(\Delta t)^3 \left.\frac{d^3x}{dt^3}\right|_t + \frac{1}{24}(\Delta t)^4 \left.\frac{d^4x}{dt^4}\right|_t + ...$$

$$x(t-\Delta t) = x(t) - \Delta t \left.\frac{dx}{dt}\right|_t + \frac{1}{2}(\Delta t)^2 \left.\frac{d^2x}{dt^2}\right|_t - \frac{1}{6}(\Delta t)^3 \left.\frac{d^3x}{dt^3}\right|_t + \frac{1}{24}(\Delta t)^4 \left.\frac{d^4x}{dt^4}\right|_t + ...$$

Therefore $$(\Delta t)^2 \left.\frac{d^2x}{dt^2}\right|_t = x(t+\Delta t) + x(t-\Delta t) - 2x(t) - \frac{1}{12}(\Delta t)^4 \left.\frac{d^4x}{dt^4}\right|_t + ...$$

$$\Rightarrow \left.\frac{d^2x}{dt^2}\right|_t \approx \frac{x(t+\Delta t) + x(t-\Delta t) - 2x(t)}{(\Delta t)^2}$$

More details
Transverse case



Net vertical force on increment of string:

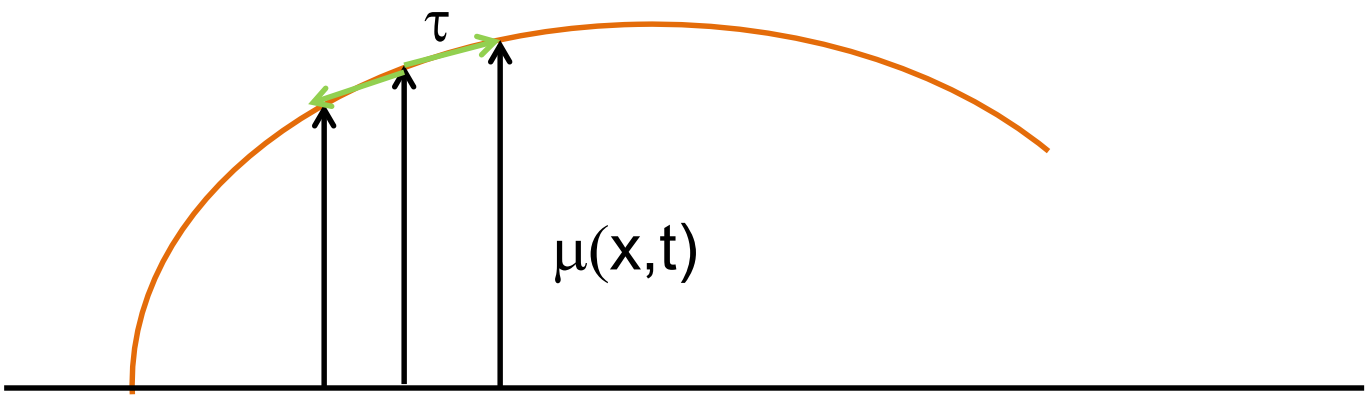
$$\tau_1 \sin\theta_1 - \tau_2 \sin\theta_2 \approx \tau_1 \tan\theta_1 - \tau_2 \tan\theta_2$$

$$\approx \tau \left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x} \right)$$

$$= \tau \Delta x \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \tau \left(\Delta x \frac{d^2 y}{dx^2} \right)$$

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

Lagrangian for continuous system:

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle:

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Euler - Lagrange equations for continuous system :

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Example :

$$L = \frac{\sigma}{2}\left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

Solution - - continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

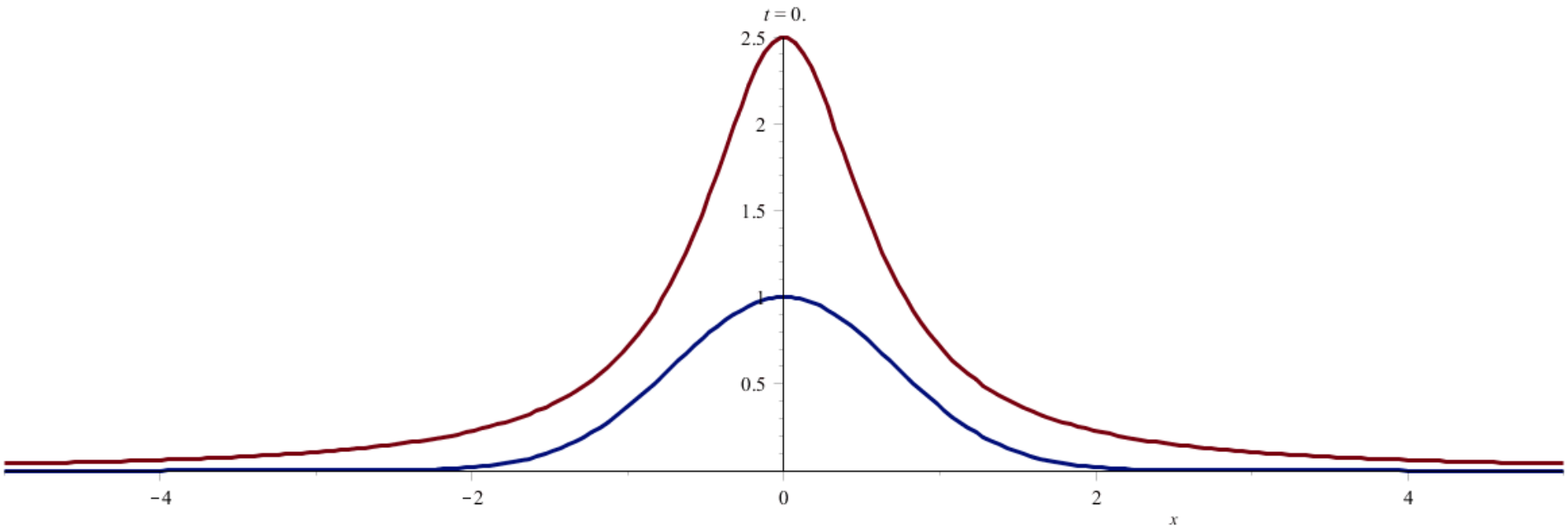
$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$

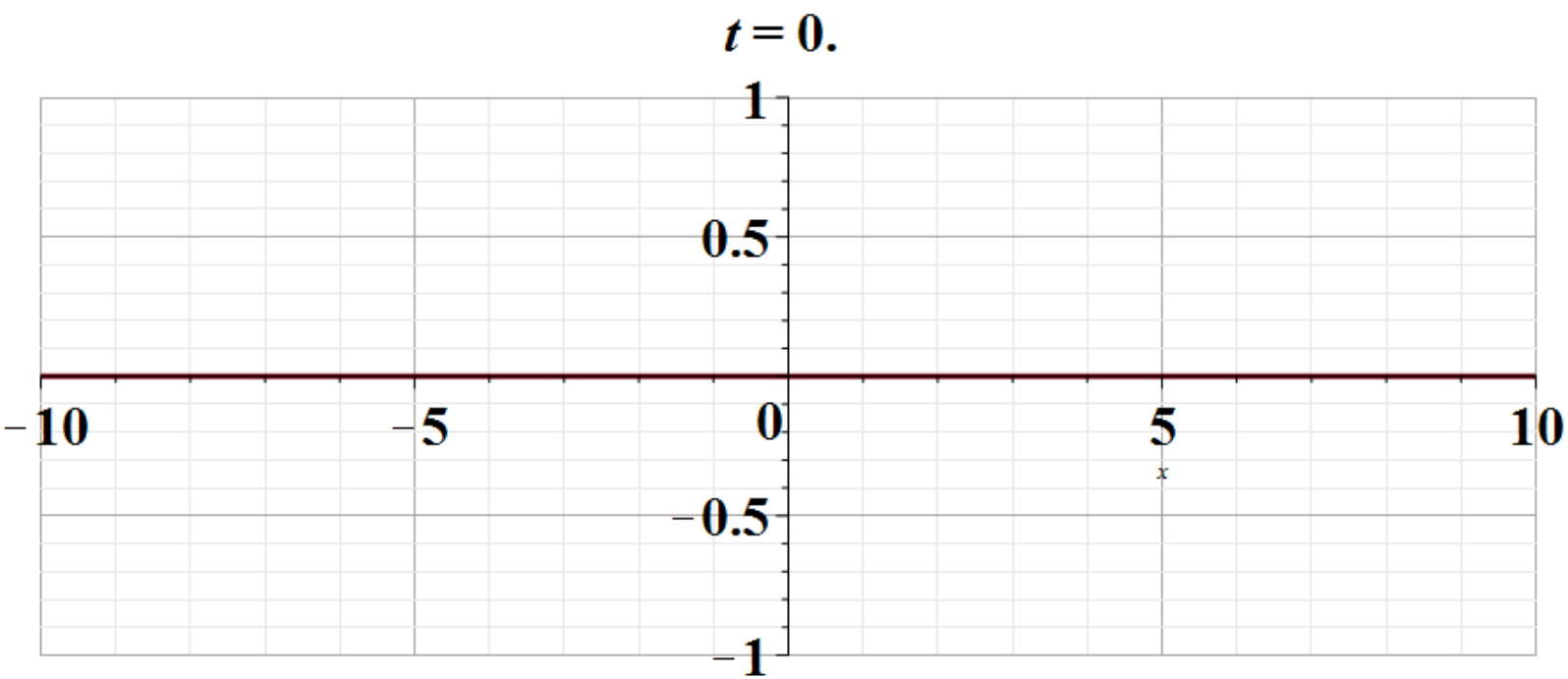


Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\dfrac{\partial \mu(x,t)}{\partial t} = -\dfrac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$



Linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$\Rightarrow Ax_1(t) + Bx_2(t)$ are also solutions (all A, B)

Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

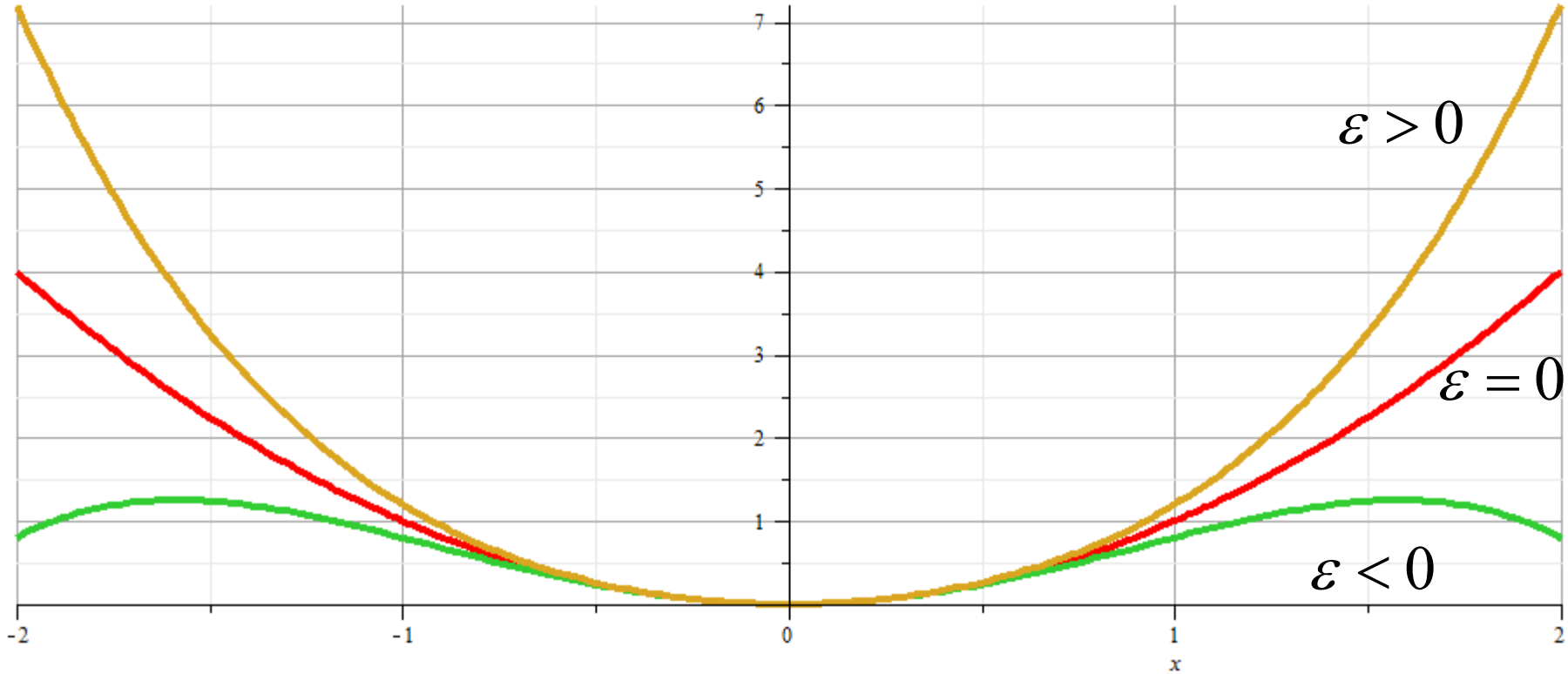
$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 \left(x + \varepsilon x^3 \right)$$

Superposition-- no longer applies

$$V(x) \approx \frac{1}{2} m\omega^2 \left(x^2 + \varepsilon x^4\right)$$



x

Non - linear example - - continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler - Lagrange equations :

zero order : $\ddot{x}_0 + \omega^2 x_0 = 0$

first order : $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Non - linear example - - continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions :

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler - Lagrange equations :

zero order : $\ddot{x}_0 + \omega^2 x_0 = 0$ $\qquad \Rightarrow x_0(t) = X_0 \cos(\omega t)$

first order : $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right)$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\}$$

$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\} + O\left(\varepsilon^2\right)$$

Non - linear example - - continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions :

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion:

$$x\left(t\right) = x_0\left(t\right) + \varepsilon x_1\left(t\right) + \ldots$$

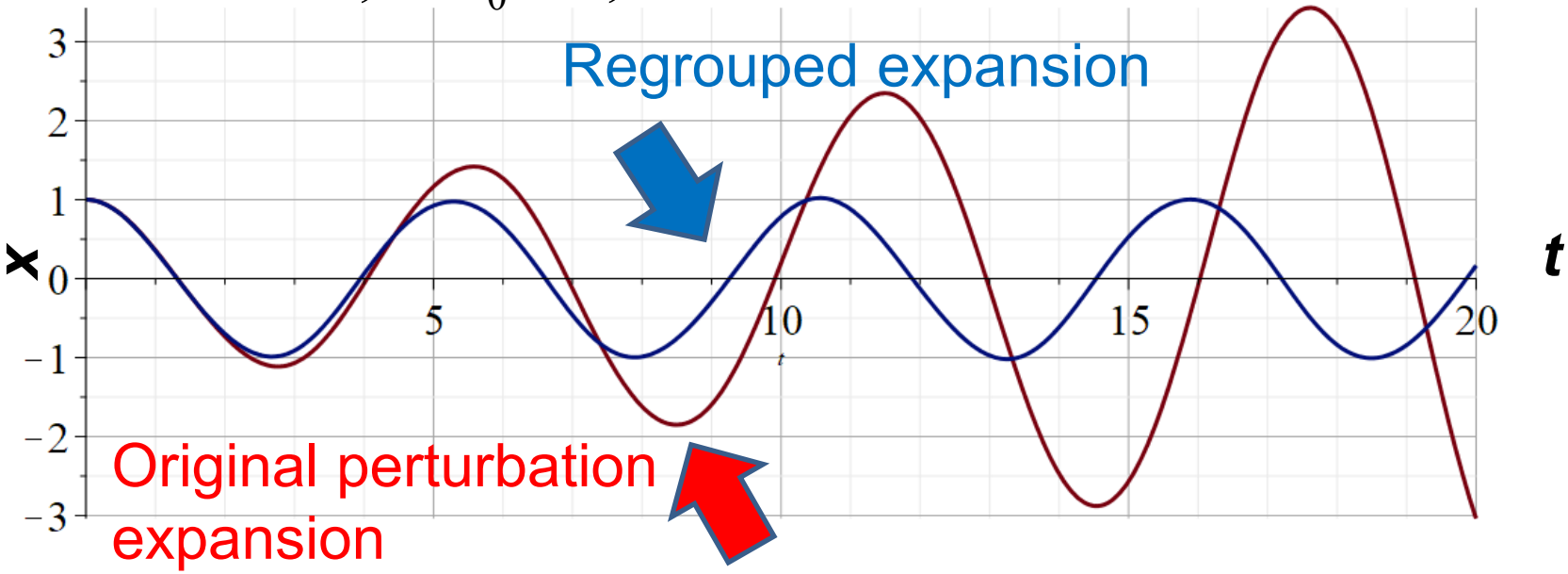
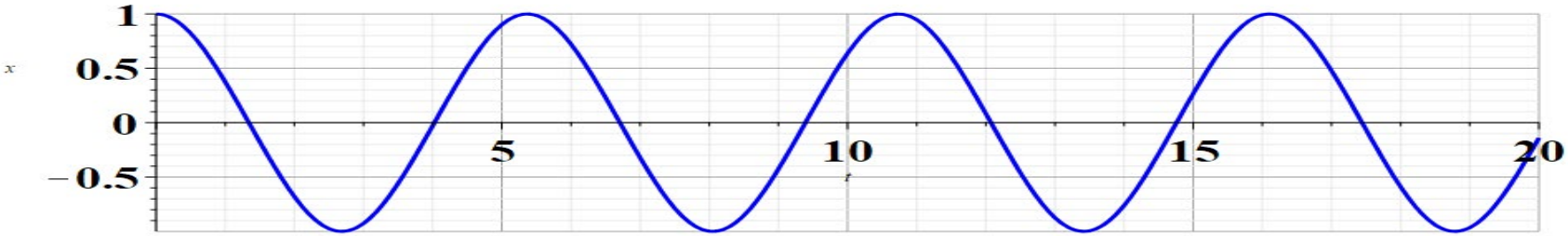
Previous result (blows up at large t):

$$x(t) = X_0 \cos\left(\omega t\right) - \varepsilon \frac{X_0^3}{8\omega^2}\left\{3\omega t\ \sin\left(\omega t\right) + \frac{1}{4}\left[\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right]\right\} + O\left(\varepsilon^2\right)$$

By rearranging terms (allowing effective frequency to vary):

$$x(t) = X_0 \cos\left(\omega\left(1 + \varepsilon \frac{3X_0^2}{8\omega}\right)t\right) - \varepsilon \frac{X_0^3}{32\omega^2}\left\{\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right\} + O\left(\varepsilon^2\right)$$

For $\omega = 1, \quad X_0 = 1, \quad \epsilon = 0.5$



Numerical solution according to Maple