PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 18: Chap. 7 (F&W)

Mechanical motion of a continuous string

1. **Masses coupled by springs ←→ masses coupled by string**
2. **Mechanics one-dimensional continuous system**
3. **The wave equation**
4. **Comments on linear vs. non-linear equations**

The one dimensional motion of a large number masses interconnected with springs provides a model of longitudinal motions of a continuous elastic spring and related topics covered in Chapter 7 of your textbook

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
17	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations		

Start reading Chapter 7. The homework problem concerns one dimensional wave motion using methods discussed in this lecture.

PHY 711 – Assignment #13

10/05/2020

Start reading Chapter 7 in **Fetter and Walecka**.

Consider a one-dimensional wave characterized by displacement $\mu(x,t)$ as a function of position x and time t is described by the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0, \tag{1}$$

where c denotes the wave speed. Find the functional form of $\mu(x,t)$ for each of these initial conditions.

1. At $t = 0$,

$$\mu(x,0) = \frac{A}{\cosh(x)} \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = 0, \tag{2}$$

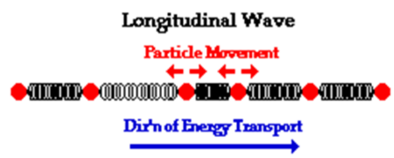
   where A is a given amplitude.

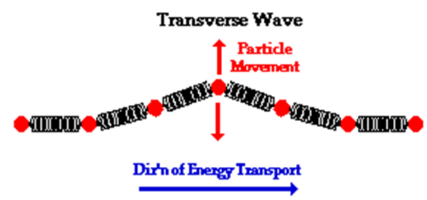
2. At $t = 0$,

$$\mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = \frac{A \sinh(x)}{\cosh^2(x)}. \tag{3}$$

Homework problem is due Wednesday.

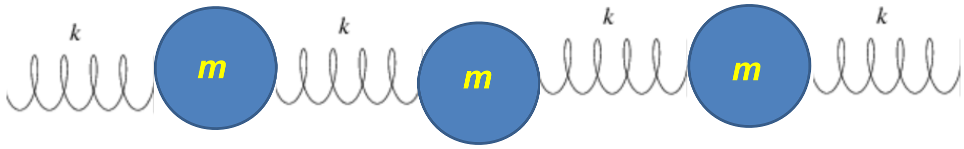
Continuum limit of masses and springs --
Longitudinal versus transverse vibrations
Images from web page:
http://www.physicsclassroom.com/class/waves/u10l1c.cfm





Visualization (taken from the web) of wave motion, showing the difference between longitudinal and transverse waves.

Longitudinal case: a system of masses and springs:



x_{i-1} $\quad$ x_i $\quad$ x_{i+1}

$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i \right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1} \right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x_i, t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

Showing how the case of the infinite mass and spring system approximates the continuous elastic string.

Discrete equation : $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation : $m\dfrac{\partial^2 \mu}{\partial t^2} = k\left(\Delta x\right)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m / \Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

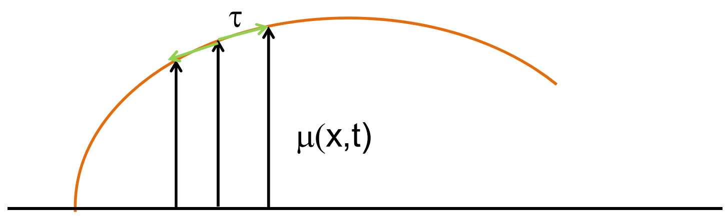
system parameter with units of (velocity)^2

For transverse oscillations on a string with tension τ and mass/length σ:

$$\left(\frac{k\Delta x}{m / \Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

Regrouping constants in terms of spring constant times increment of length and mass per unit length which combine to give a squared velocity for the longitudinal case. For the transverse case, string tension is involved.

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2\frac{\partial^2 \mu}{\partial x^2}$$

The diagram shows how the y component of the net tension contributes to the transverse motion.

Lagrangian for continuous system:

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle:

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

It is possible to adapt the Lagrangian formalism to this continuous system.

Euler - Lagrange equations for continuous system:

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Example:

$$L = \frac{\sigma}{2}\left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

The continuum version of the Euler-Lagrange equations result in the wave equation for this example.

General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

In the next several slides we will discuss solutions to the wave equation. Note that the one dimensional wave equation has some special properties.

Initial value solutions $\mu(x,t)$ to the wave equation;
attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

This method by D'Alembert is based on the special property of the wave equation.

Solution -- continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x}\psi(x')dx'\right)$$

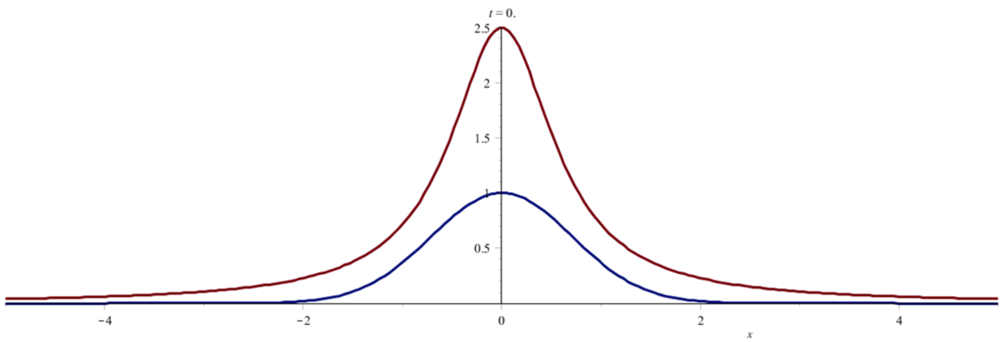
$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

D’Alembert’s method continued.

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$



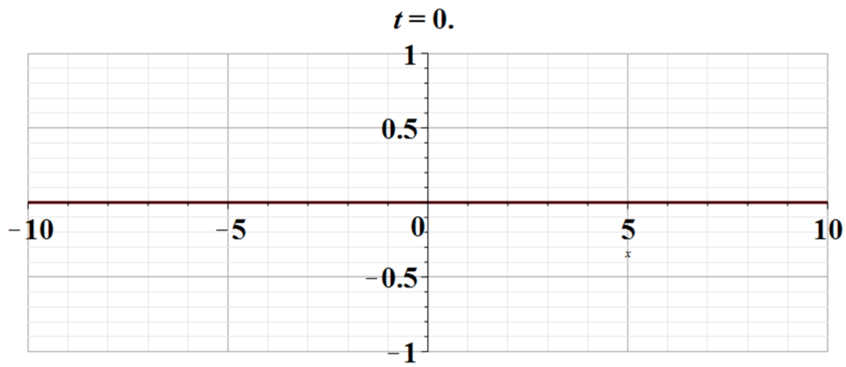
An example. (Use slide show to see animation.)

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\dfrac{\partial \mu(x,t)}{\partial t} = -\dfrac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$



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Another example. Use slide show to see animation.

Linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$\Rightarrow Ax_1(t) + Bx_2(t)$ are also solutions (all A, B)

Digression on the special properties of linear equations in contrast to complications for non-linear equations.

Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

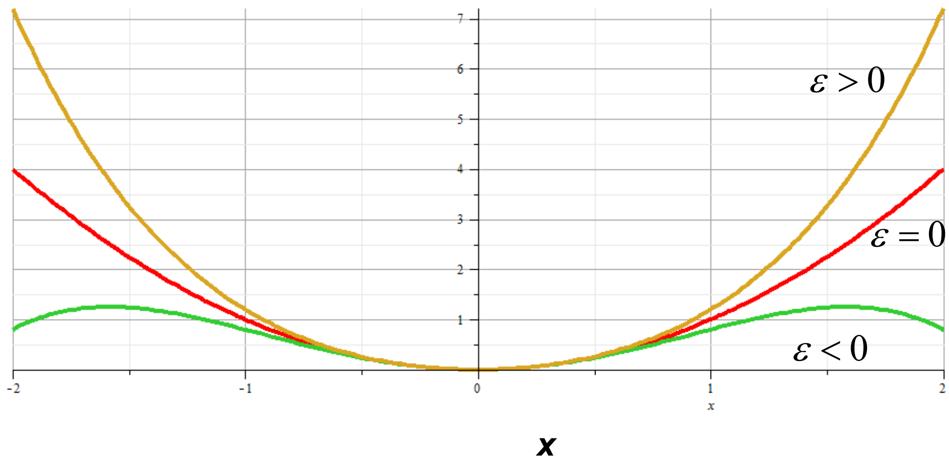
Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 \left(x + \varepsilon x^3 \right)$$

Superposition-- no longer applies

An example of the effects of non-linearity.

$$V(x) \approx \frac{1}{2} m\omega^2 \left(x^2 + \varepsilon x^4\right)$$



Plot of nonlinear potentials.

Non - linear example - - continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler - Lagrange equations :

zero order : $\ddot{x}_0 + \omega^2 x_0 = 0$

first order : $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Approximate solution to example non-linear equation.

Non - linear example - - continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions:

$x(0) = X_0 \qquad \dot{x}(0) = 0$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler - Lagrange equations:

zero order: $\ddot{x}_0 + \omega^2 x_0 = 0 \qquad \Rightarrow x_0(t) = X_0 \cos(\omega t)$

first order: $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right)$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2}\left\{3\omega t\,\sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\}$$

$$x(t) = X_0\cos(\omega t) - \varepsilon\frac{X_0^3}{8\omega^2}\left\{3\omega t\,\sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\} + O\left(\varepsilon^2\right)$$

Non-linear equation continued.

Non - linear example - - continued

$$\ddot{x}+\omega^2\left(x+\varepsilon x^3\right)=0$$

Initial conditions:

$$x(0)=X_0 \qquad \dot{x}(0)=0$$

Perturbation expansion:

$$x\left(t\right)=x_0\left(t\right)+\varepsilon x_1\left(t\right)+\ldots$$

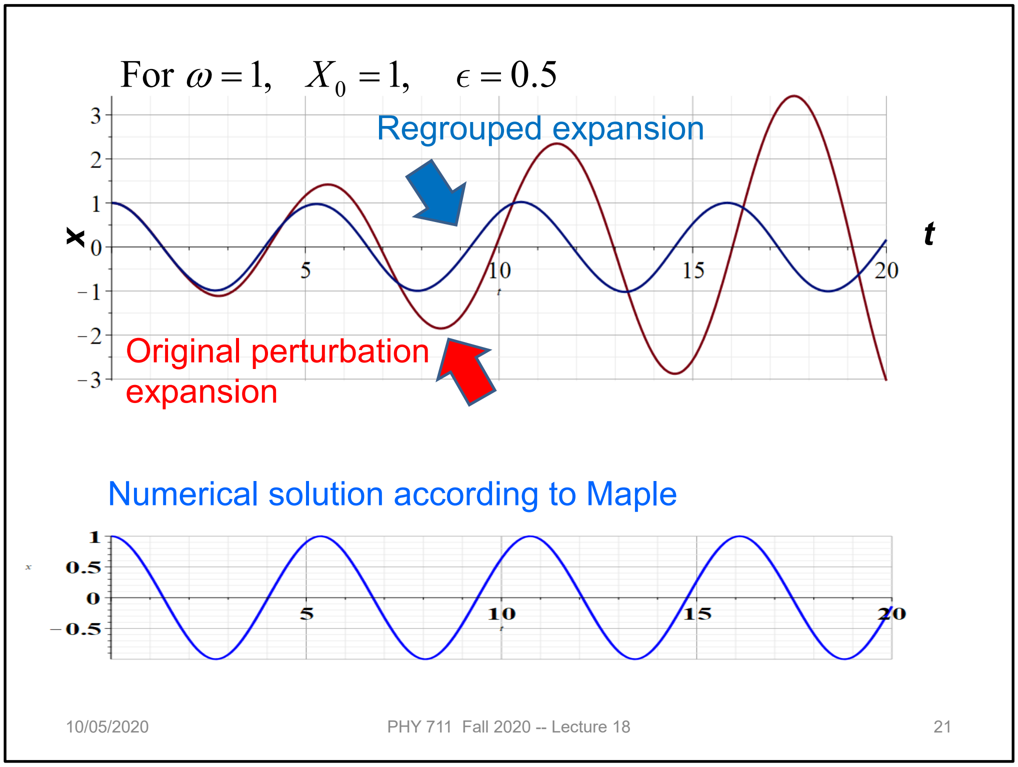
Previous result (blows up at large t):

$$x(t)=X_0\cos\left(\omega t\right)-\varepsilon\frac{X_0^3}{8\omega^2}\left\{3\omega t\ \sin\left(\omega t\right)+\frac{1}{4}\left[\cos\left(\omega t\right)-\cos\left(3\omega t\right)\right]\right\}+O\left(\varepsilon^2\right)$$

By rearranging terms (allowing effective frequency to vary):

$$x(t)=X_0\cos\left(\omega\left(1+\varepsilon\frac{3X_0^2}{8\omega}\right)t\right)-\varepsilon\frac{X_0^3}{32\omega^2}\left\{\cos\left(\omega t\right)-\cos\left(3\omega t\right)\right\}+O\left(\varepsilon^2\right)$$

More details.



Plot of results.