PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion for Lecture 19 – Chap. 7 (F&W)

Solutions of differential equations

1. The wave equation
2. Sturm-Liouville equation
3. Green's function solution methods

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		

PHY 711 -- Assignment #14

Oct. 7, 2020

Continue reading Chapter 7 in **Fetter & Walecka**.

Consider the Sturm-Liouville equation (Eq. 40.9 in F & W) with $\tau=1$, $v(x)=0$ and $\sigma=1$ for the interval $0 \leq x \leq 1$ and the boundary values $df(0)/dx=df(1)/dx=0$.

a. Find the lowest eigenvalue and the corresponding eigenfunction.
b. Choose a reasonable trial function to estimate the lowest eigenvalue and compare the estimate to the exact answer.

Next week, it is likely that we will have a take home exam instead of homework.
Perhaps distributed Monday 10/12/2020
due Monday 10/19/2020

Physics Colloquium Thursday, October 8, 2020

Online Colloquium: "Radiation-Dominated Quantum Fields in the Preinflationary Era of the Universe" — October 8, 2020 at 4 PM

Taylor Ordines
Graduate Student
Mentor, Dr. Eric Carlson
Physics Department
Wake Forest University, Winston-Salem, NC
Thursday, October 8, 2020 at 4:00 PM

Taylor Ordines recommend the following published paper from his group for topical information:
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.102.063528

Two related literature papers may also be of interest:
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.88.061501
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.95.065025

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 4 PM Wednesday
Bamidele – 7 PM Thursday
Derek – 12 PM Friday?

Your questions

From Tim –

1. What does the extra potential energy density have to do with motion on a string?When you say an applied force, is that like plucking the string or somehow putting a force on the string?

From Nick –

1. Can you elaborate on slide 14. I think I'm missing something on how Cm minimizes eps^2.

From Gao –

1. In slide 14, is Cn expression from what transformation? Similar to Fourier transformation? Thank you.

One-dimensional wave equation
representing longitudinal or transverse displacements as a function of *x* and *t* , an example of a partial differential equation --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu(x,t)}{\partial x}\right) + v(x)\mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \rho(x)\cos(\omega t + \phi)$$

Sturm-Liouville equation for spatial function:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\rho(x)}{dx}\right) + v(x)\rho(x) = \omega^2 \sigma(x)\rho(x)$$

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

given functions

applied force

solution to be determined

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: $F(x)=0$

Your question -- What does the extra potential energy density have to do with motion on a string?

Comment – In my opinion, *v(x)* has nothing to do with motion on a spring, but F & W are using the one-dimensional wave equation to motivate a more general discussion of second order differential equations. In this lecture, we will briefly review/introduce many related ideas. These will be also (and perhaps more systematically) covered in PHY 712.

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions:

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions:

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions:

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

Solution methods of Sturm-Liouville equations (assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Why all of the fuss about eigenvalues and eigenvectors?

a. They are always necessary for solving differential equations
b. Not all eigenfunctions have analytic forms.
c. It is possible to solve a differential equation without the use of eigenfunctions.
d. Eigenfunctions have some useful properties.

Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)=\lambda_m\sigma(x)f_m(x)$$

$$f_m(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)-f_n(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)$$

$$=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) = \left(\lambda_n - \lambda_m\right)\sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval

$a \leq x \leq b$:

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\,\sigma(x) f_n(x) f_m(x)$$

Vanishes for various boundary conditions
at *x=a and x=b*

Comment on orthogonality of eigenfunctions -- continued

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx}\Big|_a = \tau(x)\frac{df_m(x)}{dx}\Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx\sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

Comment on "completeness"

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x').$$

Comment on “completeness” -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

where $$C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x) \right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2\int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x) \right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m}\int_a^b dx\sigma(x)h(x)f_m(x)$$

Your question -- Can you elaborate on slide 14. I think I'm missing something on how Cm minimizes eps^2. Also -- In slide 14, is Cn expression from what transformation? Similar to Fourier transformation?

Comment – This could be similar to a Fourier transformation if the eigenfunctions $f_m(x)$ were sinusoidal (a particular choice of the Sturm-Liouville form). About the minimization of epsilon^2 – solving for the 0 of the derivative of the expression is a necessary condition for finding a minimum.

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x) \right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2\int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x) \right) f_m(x)$$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x),$ where the coefficients C_n can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h} | S | \tilde{h} \rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h} | \sigma | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

Therefore $\dfrac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle} = \dfrac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \le \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x)$ with $f_n(0) = f_n(a) = 0$

trial function $f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\frac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \frac{10}{a^2}$

Green's function solution methods -- note the following slides were note yet covered.

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Recall:

Completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right)$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

Example Sturm-Liouville problem:

$$\text{Example:} \quad \tau(x) = 1; \quad \sigma(x) = 1; \quad v(x) = 0; \quad a = 0 \quad \text{and} \quad b = L$$

$$\lambda = 1; \quad F(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right) f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

In this example:

$$\frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x - x')$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method:

$$G(x,x')=\frac{1}{W}g_a(x_<)g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x); \qquad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\ \sin(L)$$

$$\phi(x)=\phi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x \sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L \sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\phi(x)=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon} dx\delta\left(x-x'\right)$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

$$\Rightarrow W=\tau(x')\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'}=0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u} G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x} g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u} g_b(x')F(x')dx'$$