

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF online or (occasionally)
in Olin 103**

Plan for Lecture 19 – Chap. 7 (F&W)

Solutions of differential equations

- 1. The wave equation**
- 2. Sturm-Liouville equation**
- 3. Green's function solution methods**

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In this lecture, we follow the textbook to use the example of the one-dimensional wave equation to discuss ordinary differential equations more generally and develop some solution methods.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorem	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		



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The schedule continues to cover material in Chap. 7

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Oct. 7, 2020

Continue reading Chapter 7 in **Fetter & Walecka**.

Consider the Sturm-Liouville equation (Eq. 40.9 in F & W) with $\tau=1$, $v(x)=0$ and $\sigma=1$ for the interval $0 \leq x \leq 1$ and the boundary values $df(0)/dx=df(1)/dx=0$.

- Find the lowest eigenvalue and the corresponding eigenfunction.
- Choose a reasonable trial function to estimate the lowest eigenvalue and compare the estimate to the exact answer.

Next week, it is likely that we will have a take home exam instead of homework.

Perhaps distributed Monday 10/12/2020
due Monday 10/19/2020

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Homework due Friday. Note the mid term take home scheduling.

Physics Colloquium Thursday, October 8, 2020

Online Colloquium: "Radiation-Dominated Quantum
Fields in the Preinflationary Era of the Universe" —
October 8, 2020 at 4 PM

Taylor Ordines

Graduate Student

Mentor, Dr. Eric Carlson

Physics Department

Wake Forest University, Winston-Salem, NC

Thursday, October 8, 2020 at 4:00 PM

Taylor Ordines recommend the following published paper from his group
for topical information:

<https://journals.aps.org/prd/abstract/10.1103/PhysRevD.102.063528>

Two related literature papers may also be of interest:

<https://journals.aps.org/prd/abstract/10.1103/PhysRevD.88.061501>

<https://journals.aps.org/prd/abstract/10.1103/PhysRevD.95.065025>

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Colloquium this will take us out of this world. Taylor Ordines is a senior graduate student who is reporting on his project under the mentorship of Professor Eric Carlson.

One-dimensional wave equation
representing longitudinal or transverse displacements
as a function of x and t , an example of a partial
differential equation --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Review of wave equation.

The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x) \frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\tau(x) \frac{\partial \mu(x,t)}{\partial x} \right) + v(x) \mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \rho(x) \cos(\omega t + \phi)$$

Sturm-Liouville equation for spatial function:

$$-\frac{d}{dx} \left(\tau(x) \frac{d\rho(x)}{dx} \right) + v(x) \rho(x) = \omega^2 \sigma(x) \rho(x)$$

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Generalization of the wave equation. Equations in this class are separable in the time variables and the spatial variable satisfies a generalized eigenvalue problem of this form.

Linear second-order ordinary differential equations Sturm-Liouville equations

Inhomogenous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$

given functions

solution to be determined

applied force

Homogenous problem: $F(x)=0$

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We will sometimes want to generalize even further with an “inhomogeneous” term such as an applied force.

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = 0$$

Bessel functions:

$$\tau(x) = -x \quad v(x) = x \quad \sigma(x) = \frac{1}{x} \quad \lambda = \nu^2 \quad \varphi(x) = J_\nu(x)$$

Legendre functions:

$$\tau(x) = -(1-x^2) \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = l(l+1) \quad \varphi(x) = P_l(x)$$

Fourier functions:

$$\tau(x) = 1 \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = n^2 \pi^2 \quad \varphi(x) = \sin(n\pi x)$$

For now, we will focus on eigenvalues of the homogeneous equations.

Solution methods of Sturm-Liouville equations

(assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi_0(x) = 0$

Inhomogenous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$

Eigenfunctions :

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n,$

where $N_n \equiv \int_a^b \sigma(x) (f_n(x))^2 dx.$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

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The eigenfunctions of these equations have very useful properties such as completeness.

Comment on orthogonality of eigenfunctions

$$\begin{aligned}
 \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_n(x) &= \lambda_n \sigma(x) f_n(x) \\
 \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_m(x) &= \lambda_m \sigma(x) f_m(x) \\
 f_m(x) \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_n(x) - f_n(x) \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_m(x) \\
 &= (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x) \\
 -\frac{d}{dx} \left(f_m(x) \tau(x) \frac{df_n(x)}{dx} - f_n(x) \tau(x) \frac{df_m(x)}{dx} \right) &= (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x)
 \end{aligned}$$

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Orthogonality of eigenfunctions.

Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=(\lambda_n-\lambda_m)\sigma(x)f_n(x)f_m(x)$$

Now consider integrating both sides of the equation in the interval

$$a \leq x \leq b:$$

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)\bigg|_a^b=(\lambda_n-\lambda_m)\int_a^b dx\sigma(x)f_n(x)f_m(x)$$



Vanishes for various boundary conditions
at $x=a$ and $x=b$

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Orthogonality continued.

Comment on orthogonality of eigenfunctions -- continued

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) \Big|_a^b = (\lambda_n - \lambda_m) \int_a^b dx \sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

$$1. f_m(a) = f_m(b) = 0$$

$$2. \tau(x) \frac{df_m(x)}{dx} \Big|_a = \tau(x) \frac{df_m(x)}{dx} \Big|_b = 0$$

$$3. f_m(a) = f_m(b) \text{ and } \frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$$

In any of these cases, we can conclude that:

$$\int_a^b dx \sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

Orthogonality continued.

Comment on "completeness"

It can be shown that for any reasonable function $h(x)$, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where } C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a "complete" set in the sense of "spanning" the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x').$$

Notion of completeness.

Comment on “completeness” -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where } C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx \sigma(x) \left(h(x) - \sum_n C_n f_n(x) \right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2 \int_a^b dx \sigma(x) \left(h(x) - \sum_n C_n f_n(x) \right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m} \int_a^b dx \sigma(x) h(x) f_m(x)$$

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Notion of completeness and practical applications.

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the "exact" functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \quad S(x) \equiv -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The "proof" of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$$\tilde{h}(x) = \sum_n C_n f_n(x), \quad \text{where the coefficients } C_n \text{ can be}$$

assumed to be real.

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A very useful property of eigenfunctions related to homework problem

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x) \sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h} | S | \tilde{h} \rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h} | \sigma | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore } \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

Proof of theorem continued.

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{with } f_n(0) = f_n(a) = 0$

trial function $f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\frac{\langle x(a-x) | -\frac{d^2}{dx^2} | x(a-x) \rangle}{\langle x(a-x) | x(a-x) \rangle} = \frac{10}{a^2}$

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Example of the Rayleigh Ritz method.

Green's function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Completeness of eigenfunctions:

Recall:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda}$$

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The following slides present solution methods for differential equations involving the use of eigenvalues.

Solution to inhomogeneous problem by using Green's functions

Inhomogeneous problem:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$$

Green's function :

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_{\lambda}(x, x') = \delta(x - x')$$

Formal solution:

$$\varphi_{\lambda}(x) = \varphi_{\lambda 0}(x) + \int_0^L G_{\lambda}(x, x') F(x') dx'$$

Solution to homogeneous problem

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From a knowledge of the Green's function we can find solutions of related inhomogeneous equations.

Example Sturm-Liouville problem:

Example: $\tau(x) = 1$; $\sigma(x) = 1$; $v(x) = 0$; $a = 0$ and $b = L$

$$\lambda = 1; \quad F(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Example.

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x) \sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

In this example:
$$\frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x-x')$$

Solution using eigenfunctions.

Green's function :

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_{\lambda}(x, x') = \delta(x - x')$$

Green's function for the example :

$$G(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda} = \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}$$

Continued.

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

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In this case, the solution simplifies.

Alternate Green's function method :

$$G(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1 \right) g_i(x) = 0 \quad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

$$W = g_b(x) \frac{dg_a(x)}{dx} - g_a(x) \frac{dg_b(x)}{dx} = \sin(L-x) \cos(x) + \sin(x) \cos(L-x) \\ = \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L-x)}{\sin(L)} \int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx' \\ + \frac{\sin(x)}{\sin(L)} \int_x^L \sin(L-x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

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Another method of finding a Green's function.

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) g_i(x) = 0 \quad \text{for } i = a, b$$

Let

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

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Green's function based on homogeneous solutions (not eigenfunctions).

For $\epsilon \rightarrow 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x - x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} \right) \frac{1}{W} g_a(x_<) g_b(x_>) = 1$$

$$-\frac{\tau(x)}{W} \left(\frac{d}{dx} g_a(x_<) g_b(x_>) \right) \Big|_{x'-\epsilon}^{x'+\epsilon} = \frac{\tau(x')}{W} \left(g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right)$$

$$\Rightarrow W = \tau(x') \left(g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'} = 0$.

\Rightarrow Useful Green's function construction in one dimension:

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

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Some details.

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) \varphi(x) = F(x)$$

Green's function solution:

$$\begin{aligned} \varphi_\lambda(x) &= \varphi_{\lambda 0}(x) + \int_{x_l}^{x_u} G_\lambda(x, x') F(x') dx' \\ &= \varphi_{\lambda 0}(x) + \frac{g_b(x)}{W} \int_{x_l}^x g_a(x') F(x') dx' + \frac{g_a(x)}{W} \int_x^{x_u} g_b(x') F(x') dx' \end{aligned}$$

More details. To be continued.