PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 20 – Chap. 7 (F&W)

Solutions of differential equations

1. **Green's function solution methods based on eigenfunction expansions**
2. **Green's function solution methods based on solutions of the homogeneous equations**

In this lecture, we will continue our discussion of one dimensional ordinary differential equations.

4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Sturm-Liouville equations		
22	Wed, 10/14/2020	Chap. 7	Sturm-Liouville equations		

The schedule continues to cover material in Chap. 7

Plan for next week. Take home exam will be available on Monday 10/12/2020 and due Monday 10/19/2020. It is an open book/open note exam. According to the honor code, it must be your own work. You may consult with me, but NO ONE ELSE. The problems are likely to be similar to those you have had for homework. The synchronous lectures will continue through this period, but no additional homework will be assigned.

Comment on take home exam for next week.

Review – Sturm-Liouville equations defined over a range of x.

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Review of the class problems considered.

Eigenvalues and eigenfunctions of Sturm-Liouville equations

In the domain $a \leq x \leq b$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Alternative boundary conditions; 1. $f_m(a)=f_m(b)=0$

or 2. $\tau(x)\frac{df_m(x)}{dx}\bigg|_a=\tau(x)\frac{df_m(x)}{dx}\bigg|_b=0$

or 3. $f_m(a)=f_m(b)$ and $\frac{df_m(a)}{dx}=\frac{df_m(b)}{dx}$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2\,dx.$

General properties.

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x),$ where the coefficients C_n can be assumed to be real.

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Comment on the Raleigh-Ritz approximation for the lowest eigenvalues.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

Therefore $$\frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

Proof of the Rayleigh-Ritz theorem.

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

Example: $\quad -\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(0) = f_n(a) = 0$

trial function $\quad f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\frac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \frac{10}{a^2}$

Review of example from last lecture.

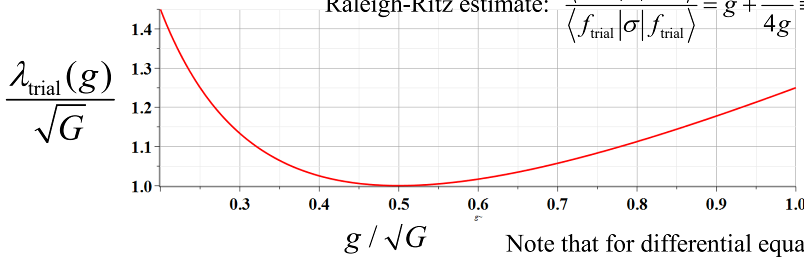
Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

Example: $-\frac{d^2 f_n(x)}{dx^2} + Gx^2 f_n(x) = \lambda_n f_n(x)$ with $f_n(-\infty) = f_n(\infty) = 0$

trial function $f_{\text{trial}}(x) = e^{-gx^2}$

Raleigh-Ritz estimate: $\frac{\left\langle f_{\text{trial}} \middle| S \middle| f_{\text{trial}} \right\rangle}{\left\langle f_{\text{trial}} \middle| \sigma \middle| f_{\text{trial}} \right\rangle} = g + \frac{G}{4g} \equiv \lambda_{\text{trial}}(g)$



$g_0 = \frac{1}{2}\sqrt{G} \qquad \lambda_{\text{trial}}(g_0) = \sqrt{G}$

Note that for differential equation of the Schoedinger equation of the harmonic oscillator:

$\sqrt{G} = \frac{m\omega}{\hbar} \qquad \lambda_{\text{trial}} = \frac{2m}{\hbar^2} E_0 \qquad \Rightarrow E_0 = \frac{\hbar\omega}{2}$

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Another example.

Recap -- Rayleigh-Ritz method of estimating the lowest eigenvalue

Example from Schroedinger equation for one-dimensional harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2 f_n(x)}{dx^2}+\frac{1}{2}m\omega^2 x^2 f_n(x)=E_n f_n(x) \qquad \text{with } f_n(-\infty)=f_n(\infty)=0$$

Trial function $\quad f_{\text{trial}}(x)=e^{-gx^2}$

Raleigh-Ritz estimate: $\dfrac{\langle f_{\text{trial}}|S|f_{\text{trial}}\rangle}{\langle f_{\text{trial}}|\sigma|f_{\text{trial}}\rangle}=\dfrac{\hbar^2}{2m}\left(g+\dfrac{m^2\omega^2/\hbar^2}{4g}\right)\equiv E_{\text{trial}}(g)$

$g_0=\dfrac{m\omega}{\hbar} \qquad E_{\text{trial}}(g_0)=\dfrac{1}{2}\hbar\omega$ ⇦ Exact answer

In this case, the minimization process yield’s the exact answer.

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_a^b G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

From a knowledge of the Green's function we can find solutions of related inhomogeneous equations.

In this lecture, we will discuss several methods of finding this Green's function.   This topic will also appear in PHY 712

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Recall: Completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right)$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

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The following slides present solution methods for differential equations involving the use of eigenvalues.

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0 \quad \text{and} \quad b=L$

$$\lambda=1; \qquad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Example.

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right) f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

In this example: $$\frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x - x')$$

Solution using eigenfunctions appropriate for this example.

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example:

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Continued.

Using Green's function to solve inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\varphi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \qquad \text{with boundary values } \varphi(0)=\varphi(L)=0$$

$$\varphi(x)=\varphi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\varphi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\varphi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

In this case, the solution simplifies.

Alternate Green's function method not based on eigenvalues but on solutions to the homogeneous problem:

$$G(x,x') = \frac{1}{W} g_a(x_<) g_b(x_>) \qquad \text{for} \quad 0 \le x \le L$$

$$\left(-\frac{d^2}{dx^2} - 1\right) g_i(x) = 0 \qquad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

$$W = g_b(x)\frac{dg_a(x)}{dx} - g_a(x)\frac{dg_b(x)}{dx} = \sin(L-x)\cos(x) + \sin(x)\cos(L-x)$$

$$= \sin(L)$$

$$\varphi(x) = \varphi_0(x) + \frac{\sin(L-x)}{\sin(L)}\int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$+ \frac{\sin(x)}{\sin(L)}\int_x^L \sin(L-x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\varphi(x) = \varphi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1}\sin\left(\frac{\pi x}{L}\right)$$

Hurray! Same as before.

Another method of finding a Green’s function.

More details on the general method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

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Green's function based on homogeneous solutions (not eigenfuntions).

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) G_\lambda(x,x') = \int_{x'-\epsilon}^{x'+\epsilon} dx \delta\left(x - x'\right)$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} \right) \frac{1}{W} g_a(x_<) g_b(x_>) = 1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx} g_a(x_<) g_b(x_>) \right)\Bigg]_{x'-\epsilon}^{x'+\epsilon} = \frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx} g_b(x') - g_b(x')\frac{d}{dx} g_a(x') \right)$$

$$\Rightarrow W = \tau(x')\left(g_a(x')\frac{d}{dx} g_b(x') - g_b(x')\frac{d}{dx} g_a(x') \right)$$

Note -- W (Wronskian) is constant, since $\dfrac{dW}{dx'} = 0$.

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

Some details.

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_a^b G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_a^x g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_x^b g_b(x')F(x')dx'$$

Note that the integral has to be performed in two parts; the method only works for one dimension.

More details.

Example --

$$\frac{d^2}{dx^2}\Phi(x) = -\rho(x)/\epsilon_0 \qquad \text{electrostatic potential for charge density } \rho(x)$$

Homogeneous equation:

$$\frac{d^2}{dx^2} g_{a,b}(x) = 0$$

Let $g_a(x) = x \qquad g_b(x) = 1$

Wronskian:

$$W = g_a(x)\frac{dg_b(x)}{dx} - g_b(x)\frac{dg_a(x)}{dx} = -1$$

Green's function:

$$G(x,x') = -x_<$$

$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0}\int_{-\infty}^{x} dx'\, x'\rho(x') + \frac{x}{\epsilon_0}\int_{x}^{\infty} dx'\,\rho(x')$$

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Another example, this time taken from electrostatics.

Example -- continued

$$\frac{d^2}{dx^2}\Phi(x) = -\rho(x)/\epsilon_0 \qquad \text{electrostatic potential for charge density } \rho(x)$$

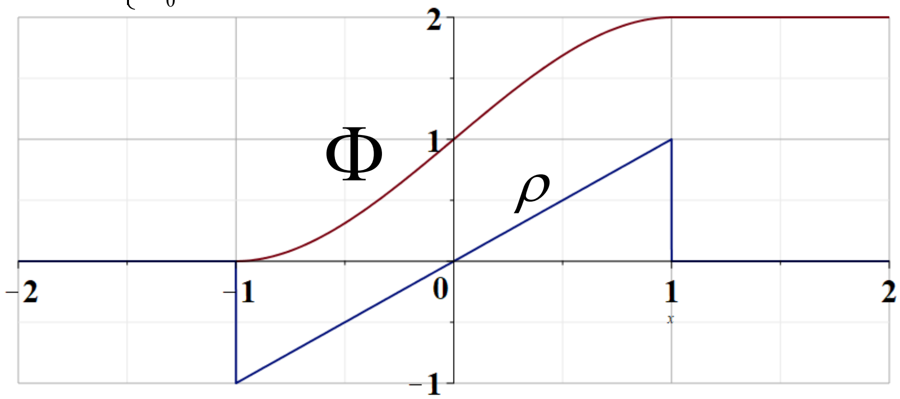
$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0}\int_{-\infty}^{x} dx'\,x'\rho(x') + \frac{x}{\epsilon_0}\int_{x}^{\infty} dx'\rho(x')$$

Suppose $\rho(x) = \begin{cases} 0 & x \le -a \\ \rho_0 x/a & -a \le x \le a \\ 0 & x \ge a \end{cases}$

$$\Phi(x) = \Phi_0(x) + \begin{cases} 0 & x \le -a \\ \dfrac{\rho_0}{\epsilon_0 a}\left(\dfrac{a^3}{3} + \dfrac{xa^2}{2} - \dfrac{x^3}{6}\right) & -a \le x \le a \\ \dfrac{2}{3\epsilon_0}\rho_0 a^2 & x \ge a \end{cases}$$

Solutions for a particular charge distribution.

$$\Phi(x) = \begin{cases} 0 & x \le -a \\ \frac{\rho_0}{\epsilon_0 a}\left(\frac{a^3}{3} + \frac{xa^2}{2} - \frac{x^3}{6}\right) & -a \le x \le a \\ \frac{2}{3\epsilon_0}\rho_0 a^2 & x \ge a \end{cases}$$



Plot of the change distribution and of the electrostatic potential.