# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM  MWF online or (occasionally in Olin 103

## Discussion for Lecture 21: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. Fourier transforms
2. Laplace transforms
3. Complex variables
4. Contour integrals

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 12 PM Friday?

# Comment on take-home exam

October 12, 2020

**PHY 711 – Mid-term Exam**

Note: This is a "take-home" exam which can be turned in any time before 5 PM (EDT) on Monday, October 19, 2020. In addition to each worked problem, please attach Maple (or Mathematica, Wolfram, etc.) work sheets as well as a full list of resources used to complete these problems. It is assumed that all work on the exam is performed under the guidelines of the honor code. In particular, you may consult with the instructor *but no one else* if you have any questions about the material. Extra points will be awarded for any corrections you identify. There are 4 problems, each of equal weight.

Note

1. Most of you will presumably email me one or several pdf files by Monday. If you prefer to turn in your written work to my mailbox, that will be fine, but you will need to let me know ahead of time.
2. You must not consult with others about this exam. However, please feel free to ask me any questions early in the exam period.
3. The problems are mostly similar to homework problems you have already completed.

**With apologies, mid term grades will not include this exam since they are due Monday at noon.**

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| 1 | Wed, 8/26/2020 | Chap. 1 | Introduction | #1 | 8/31/2020 |
| 2 | Fri, 8/28/2020 | Chap. 1 | Scattering theory | #2 | 9/02/2020 |
| 3 | Mon, 8/31/2020 | Chap. 1 | Scattering theory | #3 | 9/04/2020 |
| 4 | Wed, 9/02/2020 | Chap. 1 | Scattering theory | | |
| 5 | Fri, 9/04/2020 | Chap. 1 | Scattering theory | #4 | 9/09/2020 |
| 6 | Mon, 9/07/2020 | Chap. 2 | Non-inertial coordinate systems | | |
| 7 | Wed, 9/09/2020 | Chap. 3 | Calculus of Variation | #5 | 9/11/2020 |
| 8 | Fri, 9/11/2020 | Chap. 3 | Calculus of Variation | #6 | 9/14/2020 |
| 9 | Mon, 9/14/2020 | Chap. 3 & 6 | Lagrangian Mechanics | #7 | 9/18/2020 |
| 10 | Wed, 9/16/2020 | Chap. 3 & 6 | Lagrangian & constraints | #8 | 9/21/2020 |
| 11 | Fri, 9/18/2020 | Chap. 3 & 6 | Constants of the motion | | |
| 12 | Mon, 9/21/2020 | Chap. 3 & 6 | Hamiltonian equations of motion | #9 | 9/23/2020 |
| 13 | Wed, 9/23/2020 | Chap. 3 & 6 | Liouville theorm | #10 | 9/25/2020 |
| 14 | Fri, 9/25/2020 | Chap. 3 & 6 | Canonical transformations | | |
| 15 | Mon, 9/28/2020 | Chap. 4 | Small oscillations about equilibrium | #11 | 10/02/2020 |
| 16 | Wed, 9/30/2020 | Chap. 4 | Normal modes of vibration | #12 | 10/05/2020 |
| 17 | Fri, 10/02/2020 | Chap. 4 | Normal modes of vibration | | |
| 18 | Mon, 10/05/2020 | Chap. 7 | Motion of strings | #13 | 10/07/2020 |
| 19 | Wed, 10/07/2020 | Chap. 7 | Sturm-Liouville equations | #14 | 10/09/2020 |
| 20 | Fri, 10/09/2020 | Chap. 7 | Sturm-Liouville equations | | |
| 21 | Mon, 10/12/2020 | Chap. 7 | Fourier transforms and Laplace transforms | | |
| 22 | Wed, 10/14/2020 | Chap. 7 | Complex variables and contour integration | | |

Your questions –
From Gao –

1. What is a contour integral?

Comment – This is a useful methodology for integrating functions in the complex plane.

# Review – Sturm-Liouville equations defined over a range of x.

For $x_a \leq x \leq x_b$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Formal statement of the completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta\left(x - x'\right) \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx \sigma(x) \left(f_n(x)\right)^2$$

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \le x \le L$ and $f_n(0) = 0 = f_n(L)$

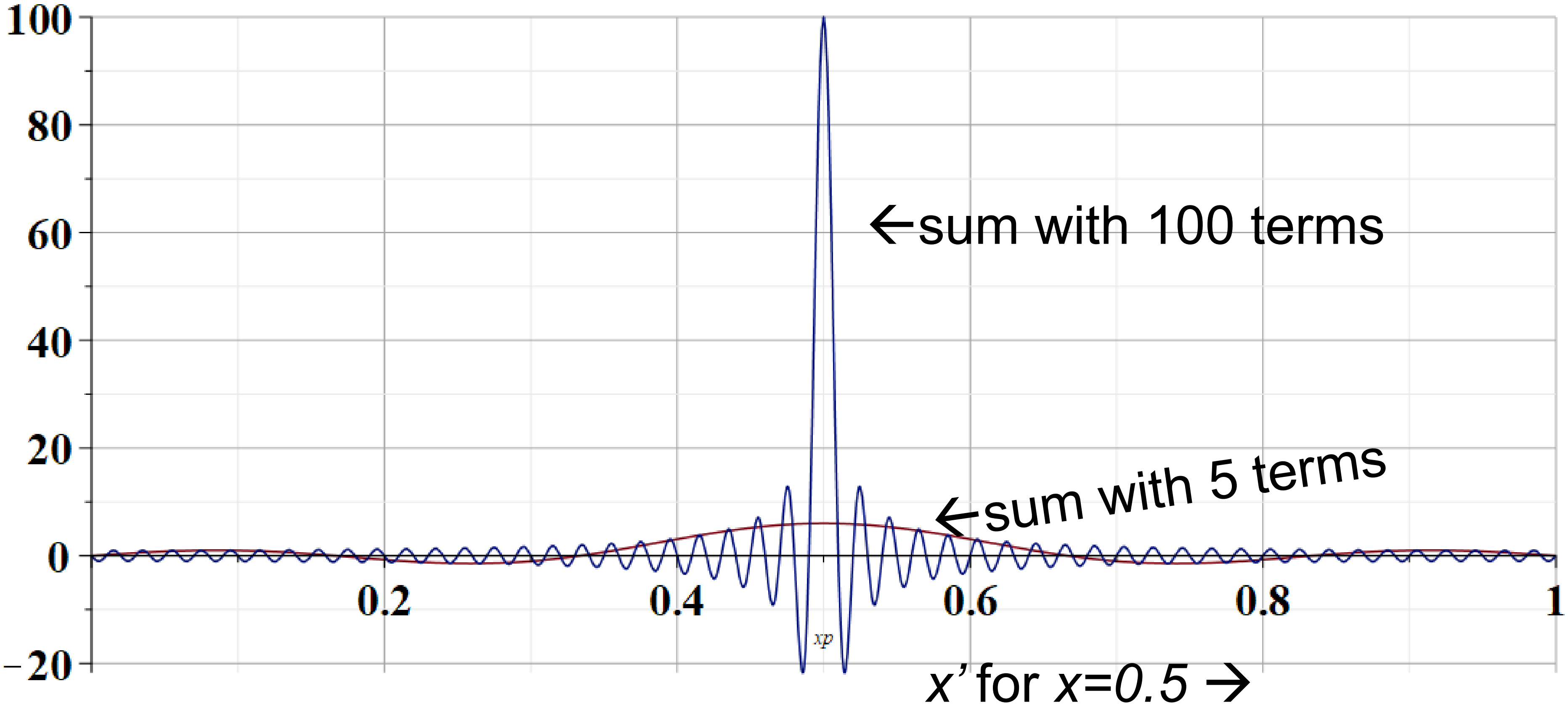
$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right) f_n(x) = \lambda_n \sigma(x) f_n(x) \quad \Rightarrow -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x$$

In this case, the normalized eigenfunctions are

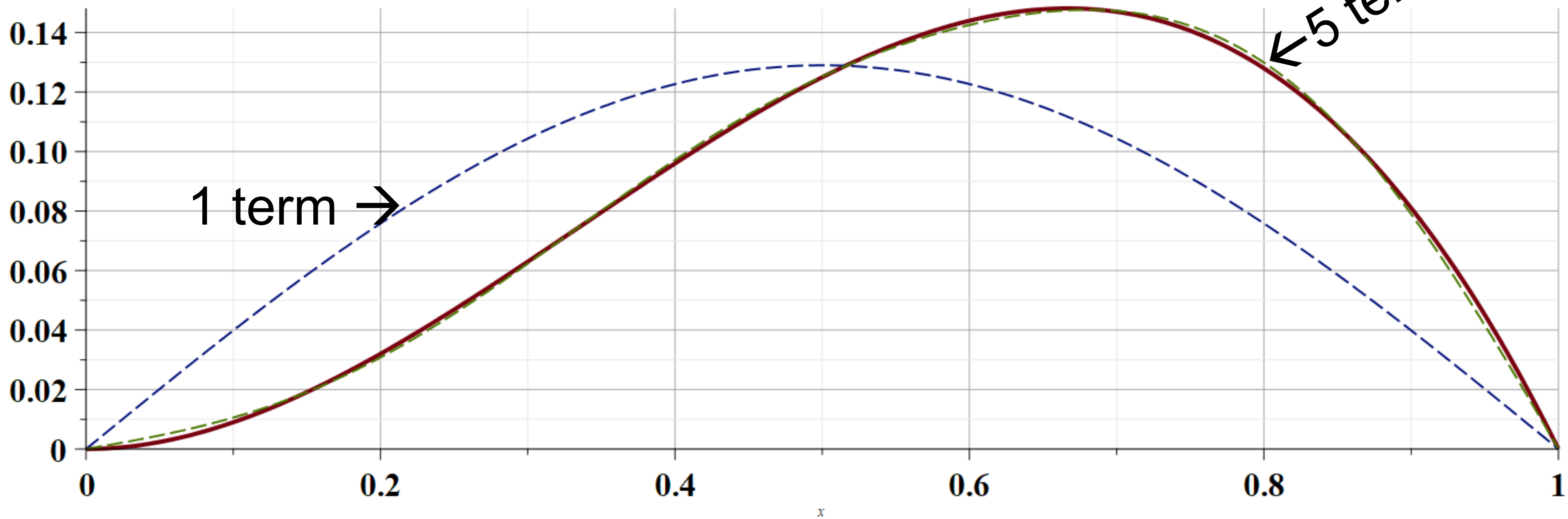
$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, .....$$

# Formal completeness for this case:

$$\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta\left(x-x'\right) \quad \text{for} \quad 0\leq x\leq L$$



Example $h(x)=x^2(1-x)$

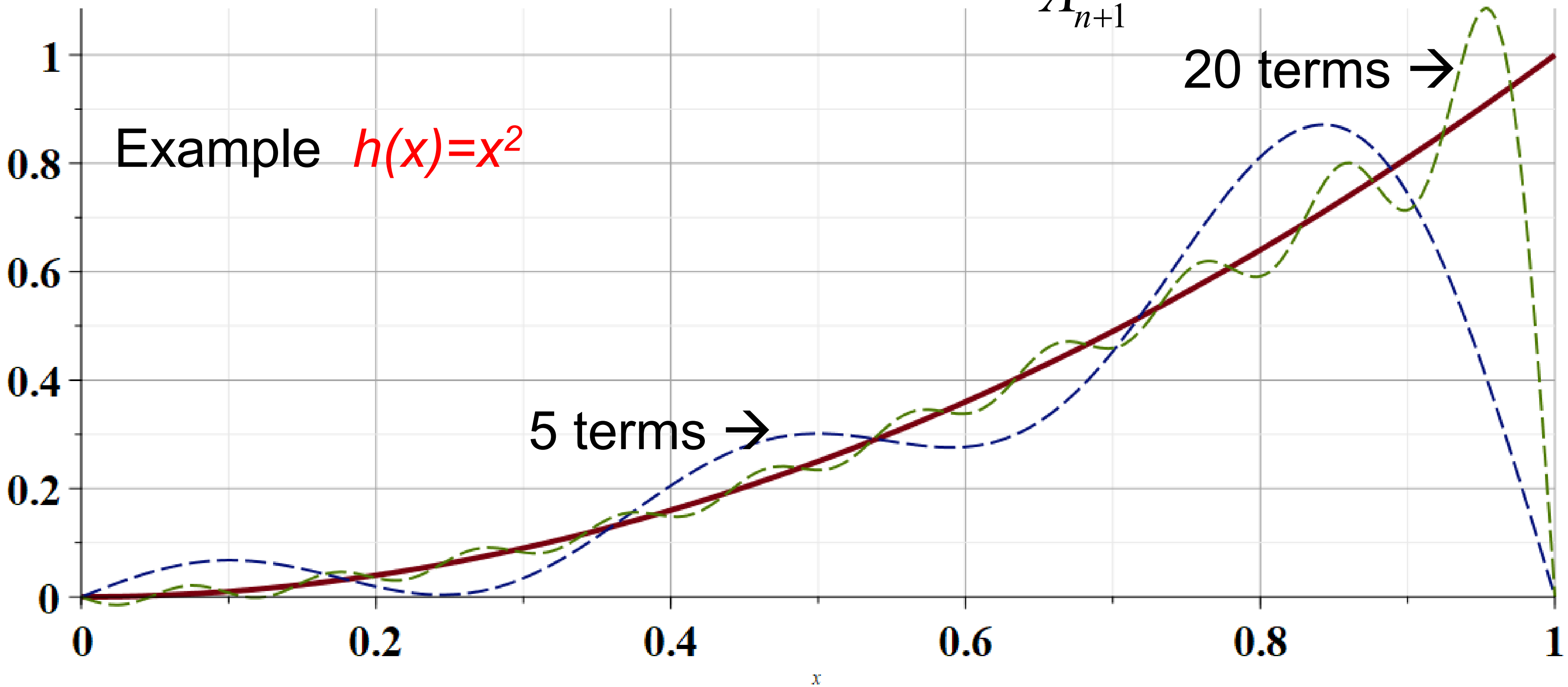



$$\tilde{h}(x,N)=\sum_{n=1}^{N}A_n\sin(n\pi x)$$

$$A_n=\int_0^1 2\sin(n\pi x')h(x')dx'=-\frac{4\left(1+2(-1)^n\right)}{n^3\pi^3}$$

# Convergence of the Fourier series

In general, $\tilde{h}(x, N \to \infty) \approx h(x)$ if $\frac{A_n}{A_{n+1}} < 1$




$$\tilde{h}(x, N) = \sum_{n=1}^{N} A_n \sin(n\pi x) \qquad A_n == -\frac{2\left(2+(-1)^n(n^2\pi^2-2)\right)}{n^3\pi^3}$$

# Using Fourier series to solve the wave equation.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

In this case, we will impose the boundary values $u(0,t) = 0 = u(L,t),$ and

the initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u(x,0)}{\partial t} = \psi(x).$

Now suppose that $u(x,t) = \rho(x)\cos(\omega t + \alpha)$ where $\omega$ and $\alpha$ are not yet known.

The spatial function $\rho(x)$ must then satisfy

$$-\frac{d^2\rho(x)}{dx^2} = \frac{\omega^2}{c^2}\rho(x) \equiv k^2\rho(x) \quad \text{with} \quad \rho(0) = \rho(L) = 0$$

We recognize this equation and find the normalized eigenfunctions to be

$$\rho_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad k_n^2 = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, ..... \qquad \omega_n = k_n c$$

Using Fourier series to solve the wave equation -- continued. The general solution can be formed by taking a linear combination of the eigenfunction results.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \alpha_n)$$

$$\text{where } \rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, ..... \quad \omega_n = \frac{n\pi}{L} c$$

The constants $C_n$ and $\alpha_n$ are determined from the initial conditions.

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty} \varphi_n \rho_n(x) \quad \text{where } \varphi_n \equiv \int_0^L \rho_n(x') \varphi(x') dx'$$

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \psi_n \rho_n(x) \quad \text{where } \psi_n \equiv \int_0^L \rho_n(x') \psi(x') dx'$$

Using Fourier series to solve the wave equation -- continued. Finding the constants from the eigenfunction (Fourier) expansion.

$$u(x,t)=\sum_{n=1}^{\infty} C_n \rho_n(x)\cos(\omega_n t+\alpha_n)$$

$$u(x,0)=\sum_{n=1}^{\infty} C_n \cos(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty} \varphi_n \rho_n(x)$$

$$\frac{\partial u(x,0)}{\partial t}=-\sum_{n=1}^{\infty} \omega_n C_n \sin(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty} \psi_n \rho_n(x)$$

Since the eigenfunctions $\rho_n(x)$ are orthogonal, the constants are immediately determined:

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n \cos(\alpha_n)\cos(\omega_n t)-C_n \sin(\alpha_n)\sin(\omega_n t)\right)\rho_n(x)$$

$$=\sum_{n=1}^{\infty}\left(\varphi_n \cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\rho_n(x)$$

Solution to wave equation from eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} \left( \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \rho_n(x)$$

where $\rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad \omega_n = \frac{n\pi}{L} c$

Recall D'Alembert's solution

$$u(x,t) = \frac{1}{2}\big(\varphi(x-ct) + \varphi(x+ct)\big) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Are these two solutions

a. Identical
b. Equivalent
c. Totally different

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.

## Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty \mathrm{e}^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \qquad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by $e^{-px}$ and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \qquad (28)$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty \mathrm{e}^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x\mathrm{e}^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{\mathrm{e}^{-\gamma x}} = \int_0^\infty \mathrm{e}^{-\gamma x}\mathrm{e}^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - \mathrm{e}^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0) = 0 \qquad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0)=0,\ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p)=-\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$=-F_0\left(\frac{\pi/L}{\left(\pi/L\right)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\displaystyle\int_0^\infty \sin(at)e^{-pt}dt=\frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x)=\frac{F_0}{\left(\pi/L\right)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Does this result look familiar?

a. Yes
b. No

# Table of Laplace transforms

## Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

| $F(s)$ | $f(t) \quad 0 \le t$ | |
|---|---|---|
| 1. $1$ | $\delta(t)$ | unit impulse at $t = 0$ |
| 2. $\frac{1}{s}$ | $1$ or $u(t)$ | unit step starting at $t = 0$ |
| 3. $\frac{1}{s^2}$ | $t \cdot u(t)$ or $t$ | ramp function |
| 4. $\frac{1}{s^n}$ | $\frac{1}{(n-1)!} t^{n-1}$ | n = positive integer |
| 5. $\frac{1}{s} e^{-as}$ | $u(t-a)$ | unit step starting at $t = a$ |
| 6. $\frac{1}{s}(1-e^{-as})$ | $u(t)-u(t-a)$ | rectangular pulse |
| 7. $\frac{1}{s+a}$ | $e^{-at}$ | exponential decay |
| 8. $\frac{1}{(s+a)^n}$ | $\frac{1}{(n-1)!} t^{n-1} e^{-at}$ | n = positive integer |
| 9. $\frac{1}{s(s+a)}$ | $\frac{1}{a}(1-e^{-at})$ | |
| 10. $\frac{1}{s(s+a)(s+b)}$ | $\frac{1}{ab}(1-\frac{b}{b-a}e^{-at}+\frac{a}{b-a}e^{-bt})$ | |
| 11. $\frac{s+\alpha}{s(s+a)(s+b)}$ | $\frac{1}{ab}[\alpha-\frac{b(\alpha-a)}{b-a}e^{-at}+\frac{a(\alpha-b)}{b-a}e^{-bt}]$ | |
| 12. $\frac{1}{(s+a)(s+b)}$ | $\frac{1}{b-a}(e^{-at}-e^{-bt})$ | |
| 13. $\frac{s}{(s+a)(s+b)}$ | $\frac{1}{a-b}(ae^{-at}-be^{-bt})$ | |

https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

In order to evaluate these integrals, we need to use complex analysis.

Check: $$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\varphi(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\varphi(u)du$$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

## Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$\left|z\right|^2 = zz^* = \left(x+iy\right)\left(x-iy\right) = x^2 + y^2$$

Polar representation

$$z = \rho\left(\cos\phi + i\sin\phi\right) = \rho\ e^{i\phi}$$

## Functions of complex variables

$$f\left(z\right) = \Re\Big(f\left(z\right)\Big) + i\Im\Big(f\left(z\right)\Big) \equiv u(x,y) + iv(x,y)$$

## Derivatives: Cauchy-Riemann equations

$$\frac{\partial f\left(z\right)}{\partial x} = \frac{\partial u\left(z\right)}{\partial x} + i\frac{\partial v\left(z\right)}{\partial x} \qquad \frac{\partial f\left(z\right)}{i\partial y} = \frac{\partial u\left(z\right)}{i\partial y} + i\frac{\partial v\left(z\right)}{i\partial y} = \frac{\partial v\left(z\right)}{\partial y} - i\frac{\partial u\left(z\right)}{\partial y}$$

Argue that $\dfrac{df}{dz} = \dfrac{\partial f\left(z\right)}{\partial x} = \dfrac{\partial f\left(z\right)}{i\partial y} \quad \Rightarrow \dfrac{\partial u\left(z\right)}{\partial x} = \dfrac{\partial v\left(z\right)}{\partial y}$ and $\dfrac{\partial v\left(z\right)}{\partial x} = -\dfrac{\partial u\left(z\right)}{\partial y}$

# Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

## Which of the following functions are analytic?

$$f(z) = e^z$$

$$f(z) = z^n$$

$$f(z) = \ln z$$

$$f(z) = z^\alpha$$

## Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = \left(x+iy\right)^2 = \left(x^2 - y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$