

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF online or (occasionally in
Olin 103**

Plan for Lecture 21: Chap. 7 & App. A-D (F&W)

**Generalization of the one dimensional wave equation →
various mathematical problems and techniques including:**

- 1. Fourier transforms**
- 2. Laplace transforms**
- 3. Complex variables**
- 4. Contour integrals**

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In this lecture we will start to cover various useful mathematical techniques.

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorem	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		



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This is the schedule. You will receive an email containing the mid term exam. It will be due next Monday.

Review – Sturm-Liouville equations defined over a range of x .

For $x_a \leq x \leq x_b$

Homogenous problem:
$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi_0(x) = 0$$

Inhomogenous problem:
$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$$

Eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a “complete” set over the range of x defined for the particular system.

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Review of the Sturm Liouville equations.

Formal statement of the completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x') \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx \sigma(x) (f_n(x))^2$$

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \leq x \leq L$ and $f_n(0) = 0 = f_n(L)$

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x) \quad \Rightarrow \quad -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x)$$

In this case, the normalized eigenfunctions are

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \dots$$

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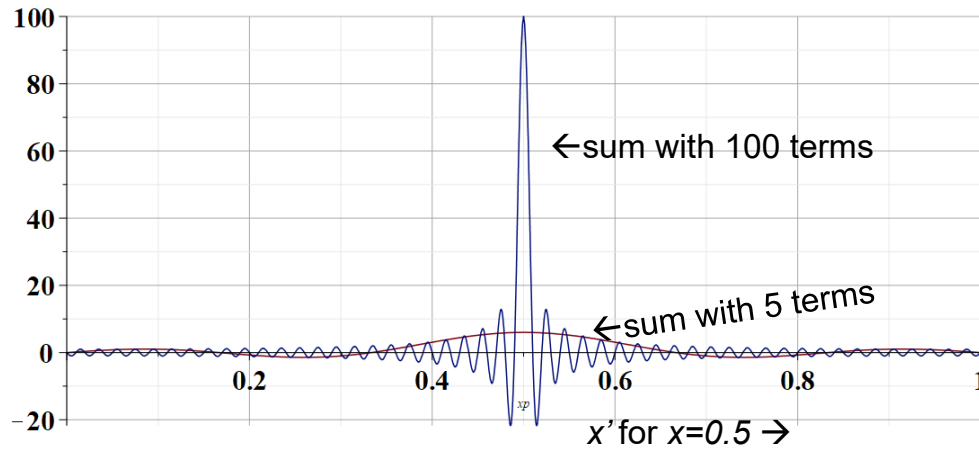
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Specializing to the simplest case.

Formal completeness for this case:

$$\frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x - x') \quad \text{for } 0 \leq x \leq L$$

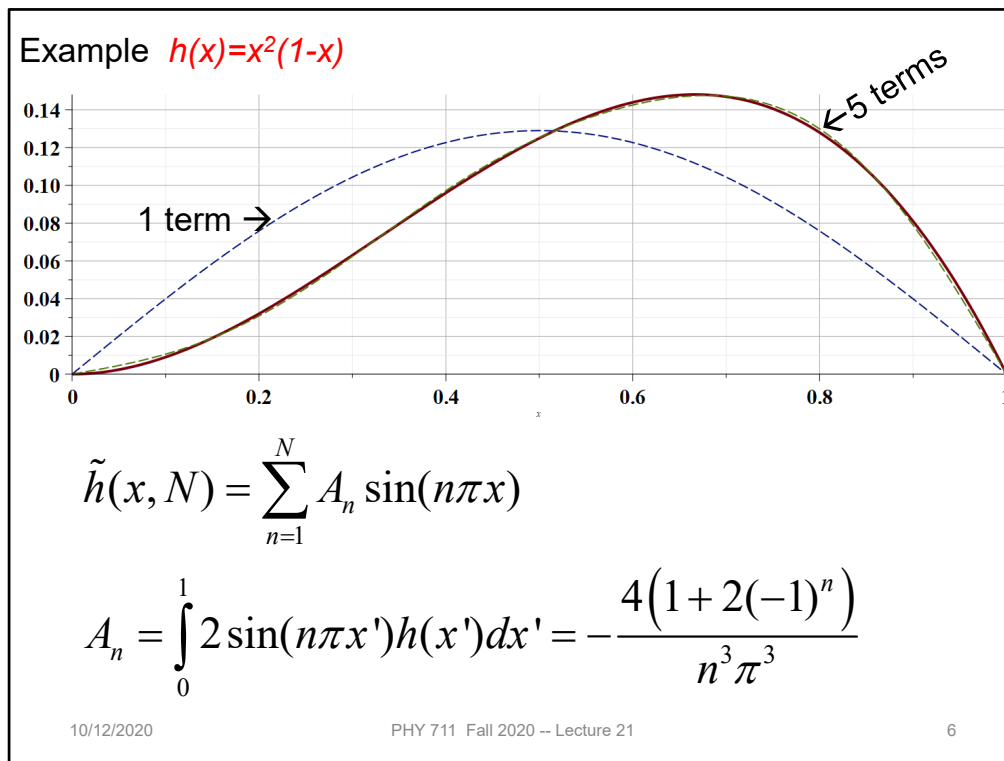


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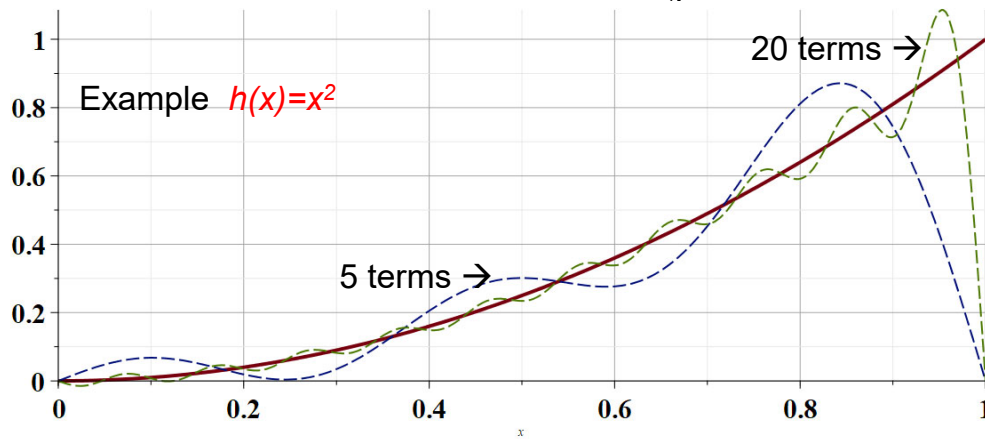
Visualizing the completeness condition.



Numerical evaluation of an example.

Convergence of the Fourier series

In general, $\tilde{h}(x, N \rightarrow \infty) \approx h(x)$ if $\frac{A_n}{n} < 1$



$$\tilde{h}(x, N) = \sum_{n=1}^N A_n \sin(n\pi x) \quad A_n = -\frac{2(2 + (-1)^n(n^2\pi^2 - 2))}{n^3\pi^3}$$

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Numerical evaluation of less convergent example.

Using Fourier series to solve the wave equation.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

In this case, we will impose the boundary values $u(0,t) = 0 = u(L,t)$, and

the initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u(x,0)}{\partial t} = \psi(x)$.

Now suppose that $u(x,t) = \rho(x)\cos(\omega t + \alpha)$ where ω and α are not yet known.

The spatial function $\rho(x)$ must then satisfy

$$-\frac{d^2 \rho(x)}{dx^2} = \frac{\omega^2}{c^2} \rho(x) \equiv k^2 \rho(x) \quad \text{with } \rho(0) = \rho(L) = 0$$

We recognize this equation and find the normalized eigenfunctions to be

$$\rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad k_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \dots \quad \omega_n = k_n c$$

Using Fourier methods to solve the wave equation.

Using Fourier series to solve the wave equation -- continued.
 The general solution can be formed by taking a linear combination of the eigenfunction results.

$$u(x, t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \alpha_n)$$

$$\text{where } \rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, \dots \quad \omega_n = \frac{n\pi}{L} c$$

The constants C_n and α_n are determined from the initial conditions.

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty} \varphi_n \rho_n(x) \quad \text{where } \varphi_n \equiv \int_0^L \rho_n(x') \varphi(x') dx'$$

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \psi_n \rho_n(x) \quad \text{where } \psi_n \equiv \int_0^L \rho_n(x') \psi(x') dx'$$

Setting the boundary and initial conditions.

Using Fourier series to solve the wave equation -- continued.
Finding the constants from the eigenfunction (Fourier) expansion.

$$u(x, t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \alpha_n)$$

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \cos(\alpha_n) \rho_n(x) = \sum_{n=1}^{\infty} \phi_n \rho_n(x)$$

$$\frac{\partial u(x, 0)}{\partial t} = -\sum_{n=1}^{\infty} \omega_n C_n \sin(\alpha_n) \rho_n(x) = \sum_{n=1}^{\infty} \psi_n \rho_n(x)$$

Since the eigenfunctions $\rho_n(x)$ are orthogonal, the constants are immediately determined:

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} (C_n \cos(\alpha_n) \cos(\omega_n t) - C_n \sin(\alpha_n) \sin(\omega_n t)) \rho_n(x) \\ &= \sum_{n=1}^{\infty} \left(\phi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \rho_n(x) \end{aligned}$$

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Determining the constants.

Solution to wave equation from eigenfunction expansion

$$u(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \rho_n(x)$$

$$\text{where } \rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad \omega_n = \frac{n\pi}{L} c$$

Recall D'Alembert's solution

$$u(x, t) = \frac{1}{2} (\varphi(x - ct) + \varphi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Are these two solutions

- a. Identical
- b. Equivalent
- c. Totally different

Interesting question.

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.

We now consider another technique that is used to solve initial value equations.

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px} \phi(x) dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \quad (26)$$

A quick introduction to Laplace transform methods.

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$. For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \quad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \quad (28)$$

An example.

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \quad (29)$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \quad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p + \gamma}. \quad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2} \left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2} \right), \quad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2} (1 - e^{-\gamma x} - \gamma x). \quad (33)$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Some details.

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0) = 0, \quad \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right) \frac{F_0}{\left(p^2 + 1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0 \left(\frac{\pi / L}{(\pi / L)^2 - 1}\right) \left(\frac{1}{p^2 + 1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2 + p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi / L)^2 - 1} \left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L} \sin(x) \right)$$

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More details.

Table of Laplace transforms

Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

$F(s)$	$f(t)$ $0 \leq t$
1. 1	$\delta(t)$ unit impulse at $t = 0$
2. $\frac{1}{s}$	1 or $u(t)$ unit step starting at $t = 0$
3. $\frac{1}{s^2}$	$t \cdot u(t)$ or t ramp function
4. $\frac{1}{s^n}$	$\frac{1}{(n-1)!} t^{n-1}$ $n = \text{positive integer}$
5. $\frac{1}{s} e^{-as}$	$u(t - a)$ unit step starting at $t = a$
6. $\frac{1}{s} (1 - e^{-as})$	$u(t) - u(t - a)$ rectangular pulse
7. $\frac{1}{s + a}$	e^{-at} exponential decay
8. $\frac{1}{(s + a)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$ $n = \text{positive integer}$
9. $\frac{1}{s(s + a)}$	$\frac{1}{a} (1 - e^{-at})$
10. $\frac{1}{s(s + a)(s + b)}$	$\frac{1}{ab} (1 - \frac{b}{b-a} e^{-at} + \frac{a}{b-a} e^{-bt})$
11. $\frac{s + \alpha}{s(s + a)(s + b)}$	$\frac{1}{ab} [\alpha - \frac{b(\alpha - a)}{b-a} e^{-at} + \frac{a(\alpha - b)}{b-a} e^{-bt}]$
12. $\frac{1}{(s + a)(s + b)}$	$\frac{1}{b-a} (e^{-at} - e^{-bt})$
13. $\frac{s}{(s + a)(s + b)}$	$\frac{1}{a-b} (ae^{-at} - be^{-bt})$

<https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf>

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Table of transforms for simple functions.

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt} \phi(t) dt$$

In order to evaluate these integrals, we need to use complex analysis.

$$\phi(t) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_\phi(p) dp$$

$$\text{Check: } \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_\phi(p) dp = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} dp \int_0^\infty e^{-pu} \phi(u) du$$

$$\begin{aligned} \frac{1}{2\pi i} \int_0^\infty \phi(u) du \int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)} dp &= \frac{1}{2\pi i} \int_0^\infty \phi(u) du \int_{-\infty}^\infty e^{\lambda(t-u)} e^{is(t-u)} i ds \\ &= \frac{1}{2\pi i} \int_0^\infty \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \delta(t-u) \right) \\ &= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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Mathematical treatment of general case.

Complex numbers

$$i \equiv \sqrt{-1} \quad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos \phi + i \sin \phi) = \rho e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x, y) + iv(x, y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i \frac{\partial v(z)}{\partial x} \quad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i \frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i \frac{\partial u(z)}{\partial y}$$

$$\text{Argue that } \frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \quad \text{and} \quad \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$$

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Introduction to properties of complex numbers.

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$$f(z) = e^z$$

$$f(z) = z^n$$

$$f(z) = \ln z$$

$$f(z) = z^\alpha$$

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Notion of analytic function. Some of these functions are not analytic.

Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

Some details. To be continued.