PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally in Olin 103

Discussion for Lecture 22: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Fourier transforms**
2. **Laplace transforms**
3. **Complex variables**
4. **Contour integrals**

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		

https://www.physics.wfu.edu/wfu-phy-news/seminars-2020-fall/

Online Colloquium: "Transition Metal Dichalcogenides: An Overview of Their Synthesis, Properties and Future Applications" — October 15, 2020 at 4 PM

Gabriel Marcus

Graduate Student

Mentor, Dr. David Carroll

Physics Department

Center for Nanotechnology and Molecular Materials

Wake Forest University, Winston-Salem, NC

Thursday, October 15, 2020 at 4:00 PM

Via Video Conference (contact wfuphys@wfu.edu for link information)

https://www.sciencedirect.com/science/article/pii/S2211285516301458 (Metallic 1T phase MoS2 nanosheets for high-performance thermoelectric energy harvesting)

https://onlinelibrary.wiley.com/doi/full/10.1002/adma.201700070 (2D Chalcogenide Nanoplate Assemblies for Thermoelectric Applications)

https://onlinelibrary.wiley.com/doi/full/10.1002/adma.201702968 (Self-Assembled Heterostructures: Selective Growth of Metallic Nanoparticles on V2–VI3 Nanoplates)

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Jeanette – 11 AM Wednesday
Bamidele – 7 PM Thursday
Zhi– ??
Derek – 12 PM Friday

Your questions –

From Jeanette –

1. Slides 7 + 8 - I followed the examples of non-analytic functions, but I'm not sure how the contour integral relates.

From Tim –

1. Could we go over briefly how you solved 1+z^4 on page 10?

From Gao –

1. Could you explain more about the residue theorem?

Introduction to complex variables

1. Basic properties
2. Notion of an analytic complex function
3. Cauchy integral theory
4. Analytic functions and functions with poles
5. Evaluating integrals of functions in the complex plane

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Examples of analytic functions

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = (x+iy)^2 = \left(x^2 - y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

Examples of non-analytic functions

$$\text{Note that } z = \rho e^{i\phi} = \rho e^{i\phi + i2\pi n} \quad \text{for any integer } n$$

$$\Rightarrow \ln z = \ln \rho + i\left(\phi + 2\pi n\right)$$

$\ln z$ is not analytic because it is multivalued

$$\Rightarrow z^{\alpha} = \rho^{\alpha} e^{i\alpha\phi} e^{i2\pi n\alpha}$$

z^{α} is not analytic for non-integer α because it is multivalued

Behavior of $f(z) = \dfrac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Behavior of $f(z)=\frac{1}{z^n}$ about the point $z=0$:

For an integer n, consider

$$\oint \frac{1}{z^n}dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n}\int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n\neq 1 \\ 2\pi i & n=1 \end{cases}$$

This observation helps us to focus on a special kind of singularity called a "pole"

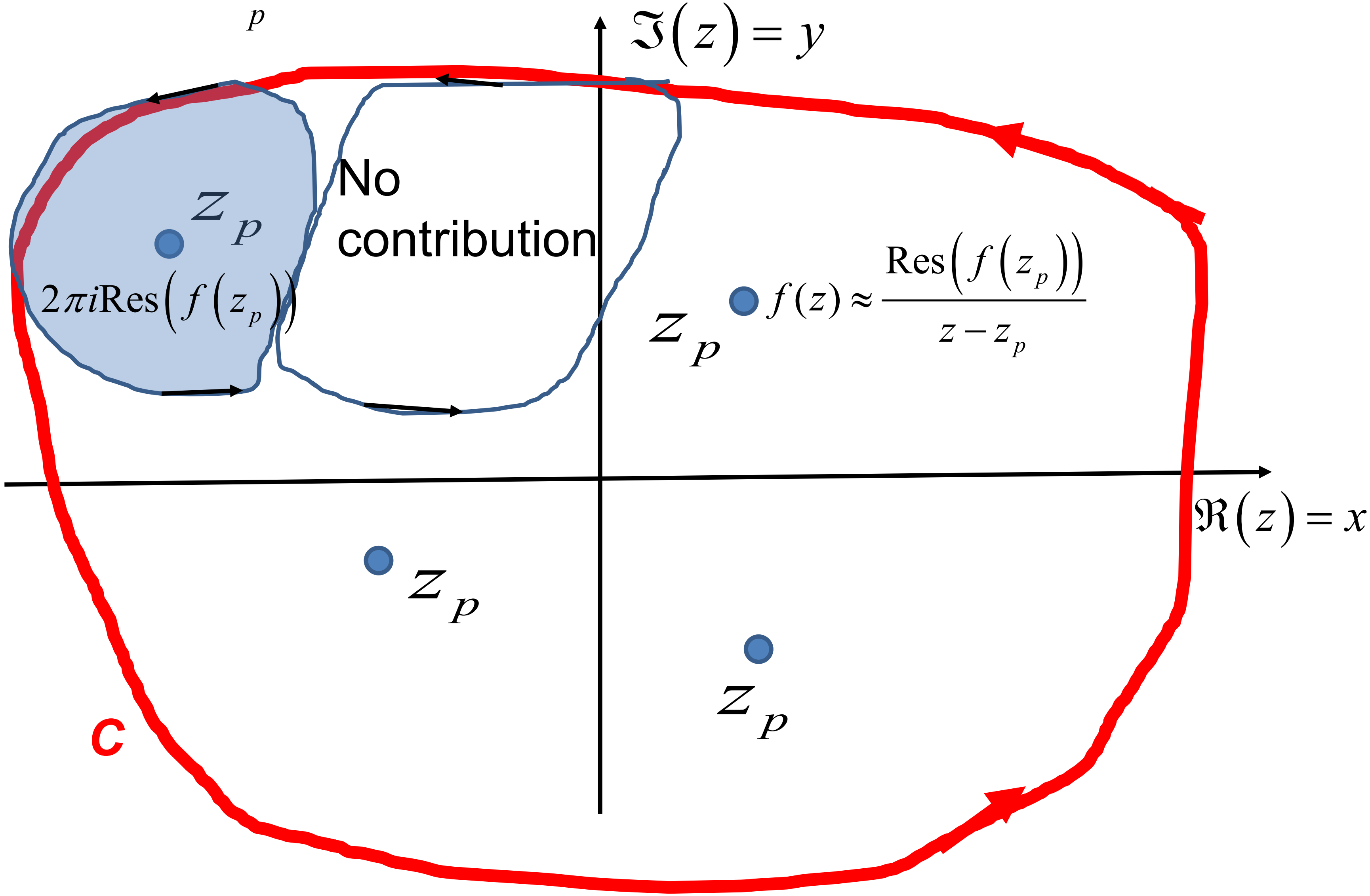
For $f(z)$ in the vicinity of $z=z_p$: $f(z)\approx \frac{g(z_p)}{z-z_p}$

Therefore: $\oint f(z)dz = 0$ or $\oint f(z)dz = g(z_p)\oint \frac{dz}{z-z_p} = 2\pi i g(z_p)$

Integration does not include z_p

Integration does include z_p

$$\oint_C f(z)dz = 2\pi i \sum_p \mathrm{Res}\left(f(z_p)\right)$$



General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{h(z)}{\left(z-z_p\right)^m} \underset{z\to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

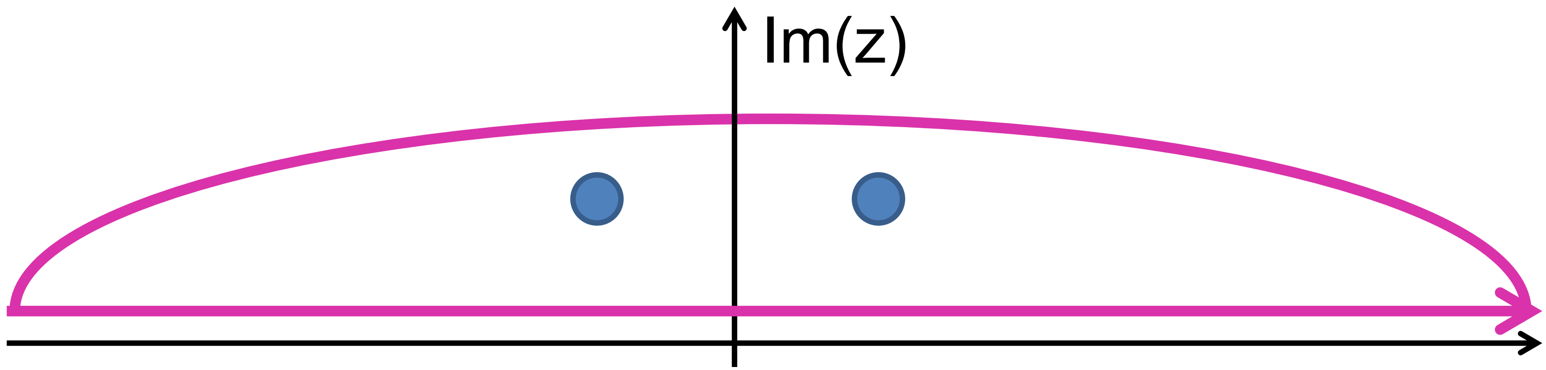
Since $h(z) \equiv \left(z-z_p\right)^m f(z)$ is analytic near z_p, we can make a Taylor exansion

about z_p: $\quad h(z) \approx h(z_p) + \left(z-z_p\right)\frac{dh(z_p)}{dz} + \ldots + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}h(z_p)}{dz^{m-1}} + \ldots.$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$

In the following examples *m=1*

Example: $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx + 0 = \oint \frac{z^2}{1+z^4}dz$$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\mathrm{Res}\left(z_p = e^{i\pi/4}\right) + \mathrm{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

Note:
m=1

$$\mathrm{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \mathrm{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{2}\left(\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) - \left(-\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Some details:

Note that: $e^{i\pi} = -1 = e^{-i\pi}$

$$e^{-3i\pi/4} = e^{i\pi/4 - i\pi} = -e^{i\pi/4}$$

$$f(z) = \frac{z^2}{1+z^4}$$

$$\text{Res}\left(f(z = e^{i\pi/4}\right) = \frac{\left(e^{i\pi/4}\right)^2}{\left(e^{i\pi/4} - e^{3i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-3i\pi/4}\right)}$$

$$= \frac{e^{i\pi/2}}{\left(e^{i\pi/4} + e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} + e^{i\pi/4}\right)}$$

$$= \frac{e^{i\pi/4}}{2\left(i - (-i)\right)} = \frac{e^{i\pi/4}}{4i}$$

Question – Could we have chosen the contour in the lower half plane?

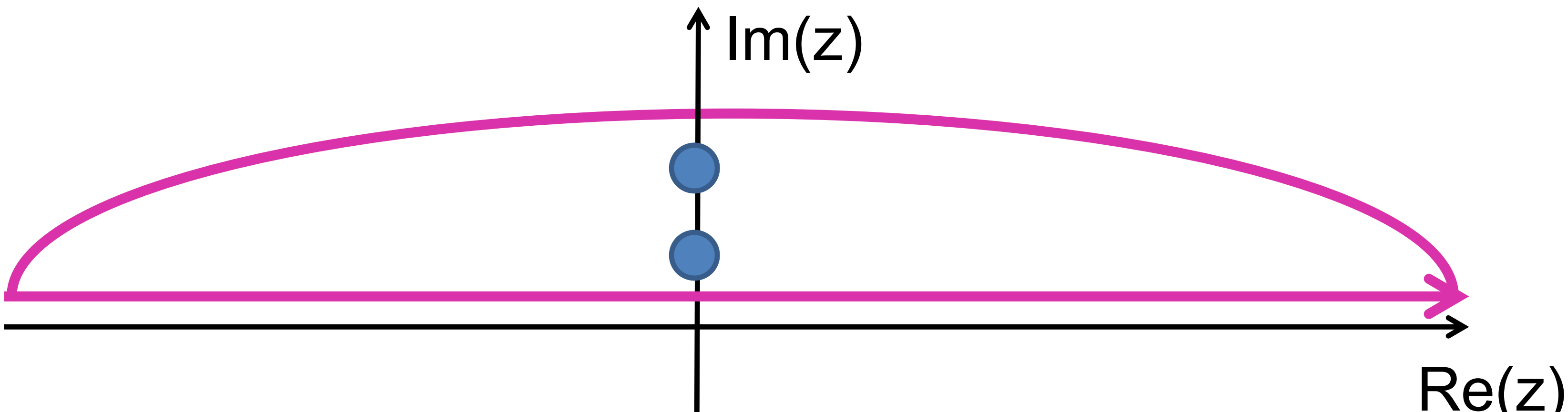
a. Yes b. No

Another example: $I = \int_0^\infty \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx$.

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$4z^4+5z^2+1 = 4(z-i)\left(z-\tfrac{i}{2}\right)(z+i)\left(z+\tfrac{i}{2}\right)$$

Note:
m=1



$$I = 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$= 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$= \frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

Question – Could we have chosen the contour in the lower half plane?

a. Yes b. No

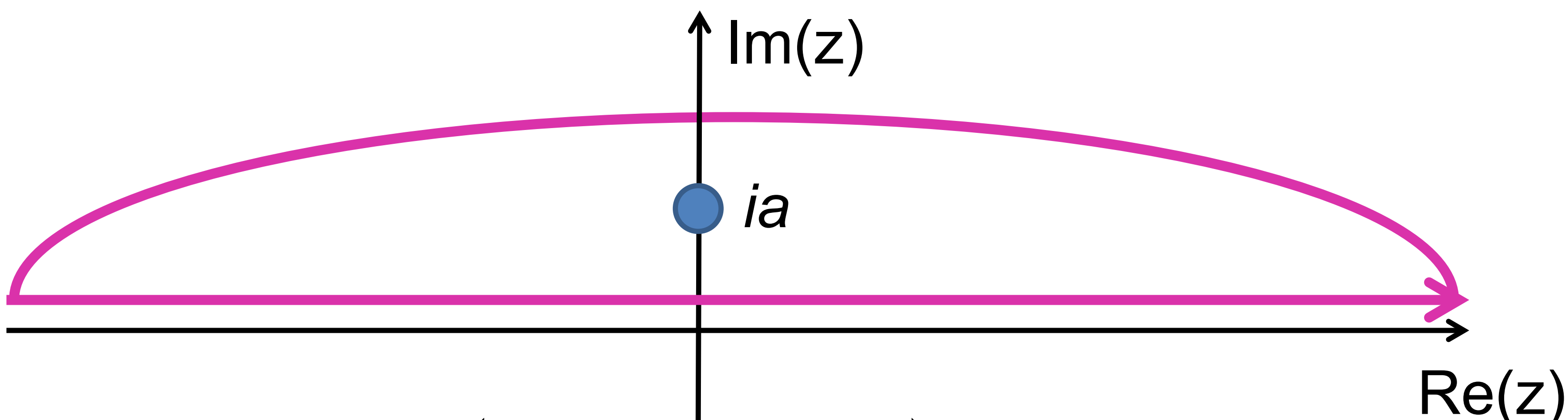
Note that for $a > 0$ and $z_I > 0$

in the lower half plane: $e^{iaz} = e^{iaz_R}e^{az_I}$

Another example: $I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx \quad \text{for } k > 0 \text{ and } a > 0$

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$



$$I = 2\pi i \left(\text{Res}\left(z_p = ia \right) \right) = \pi e^{-ka}$$

Some details --

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$

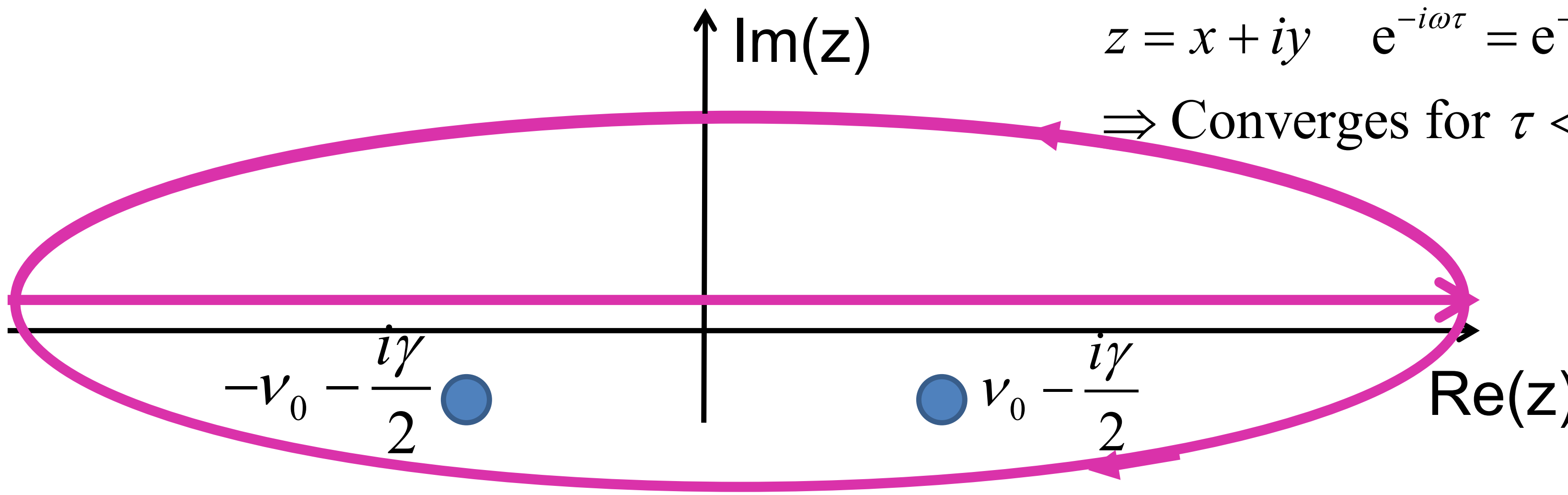
$$\frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz = 2\pi i \frac{1}{i} \lim_{z \to ia} \left((z - ia) \frac{z e^{ikz}}{z^2 + a^2} \right)$$

$$= 2\pi i \frac{1}{i} \frac{ia e^{-ka}}{2ia} = \pi e^{-ka}$$

From the Drude model of dielectric response --

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

where ω_p, ω_0, and γ are positive constants



Upper hemisphere:

$z = x + iy \quad e^{-i\omega\tau} = e^{-ix\tau + y\tau}$

$\Rightarrow$ Converges for $\tau < 0$

Lower hemisphere:

$z = x - iy \quad e^{-i\omega\tau} = e^{-ix\tau - y\tau}$

$\Rightarrow$ Converges for $\tau > 0$

$$\nu_0 \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

From the Drude model of dielectric response -- continued --

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

$$G(\tau) = \omega_p^2 \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-\gamma\tau/2} \dfrac{\sin \nu_0 \tau}{\nu_0} & \text{for } \tau > 0 \end{cases}$$

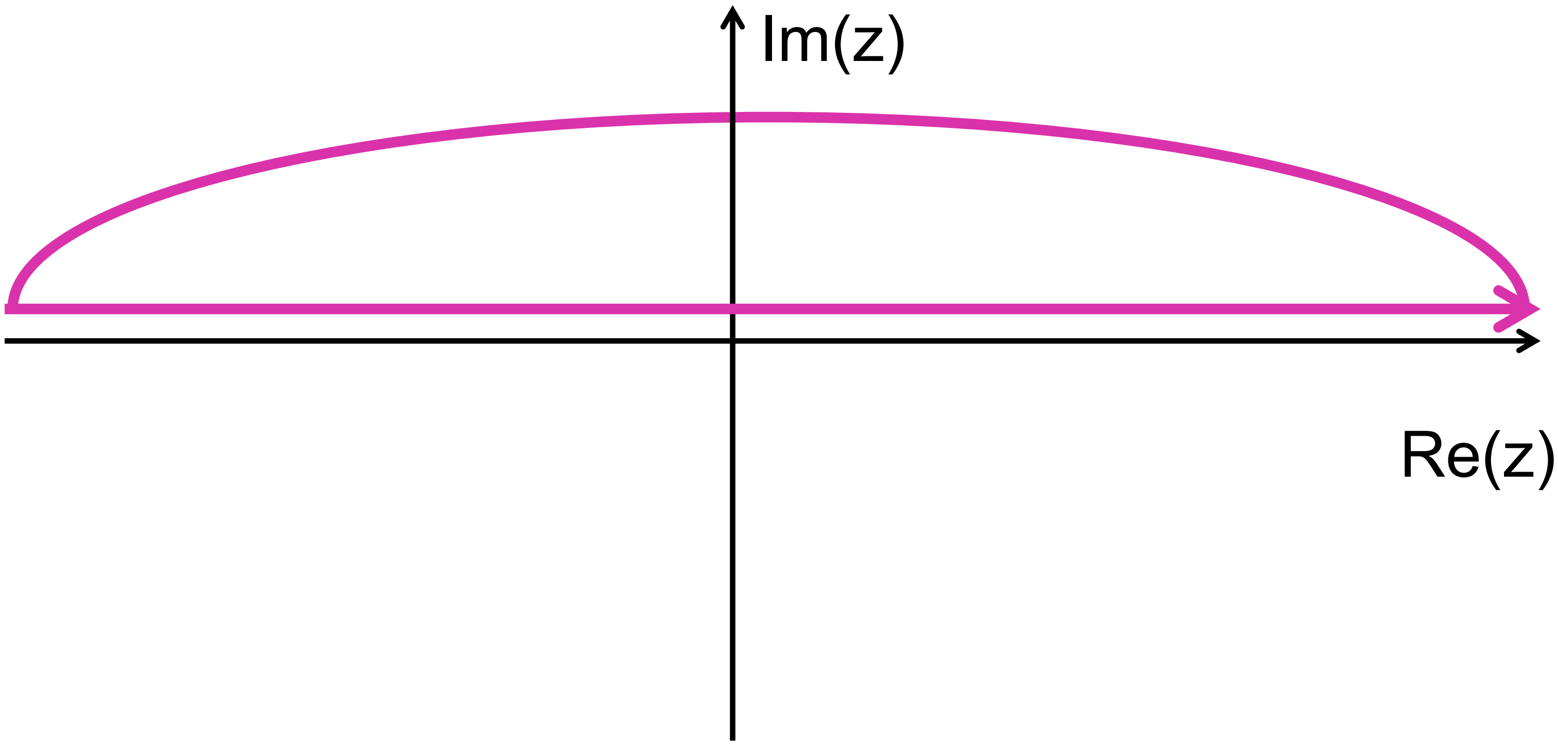
Cauchy integral theorem for analytic function *f(z)*:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'.$$

Example

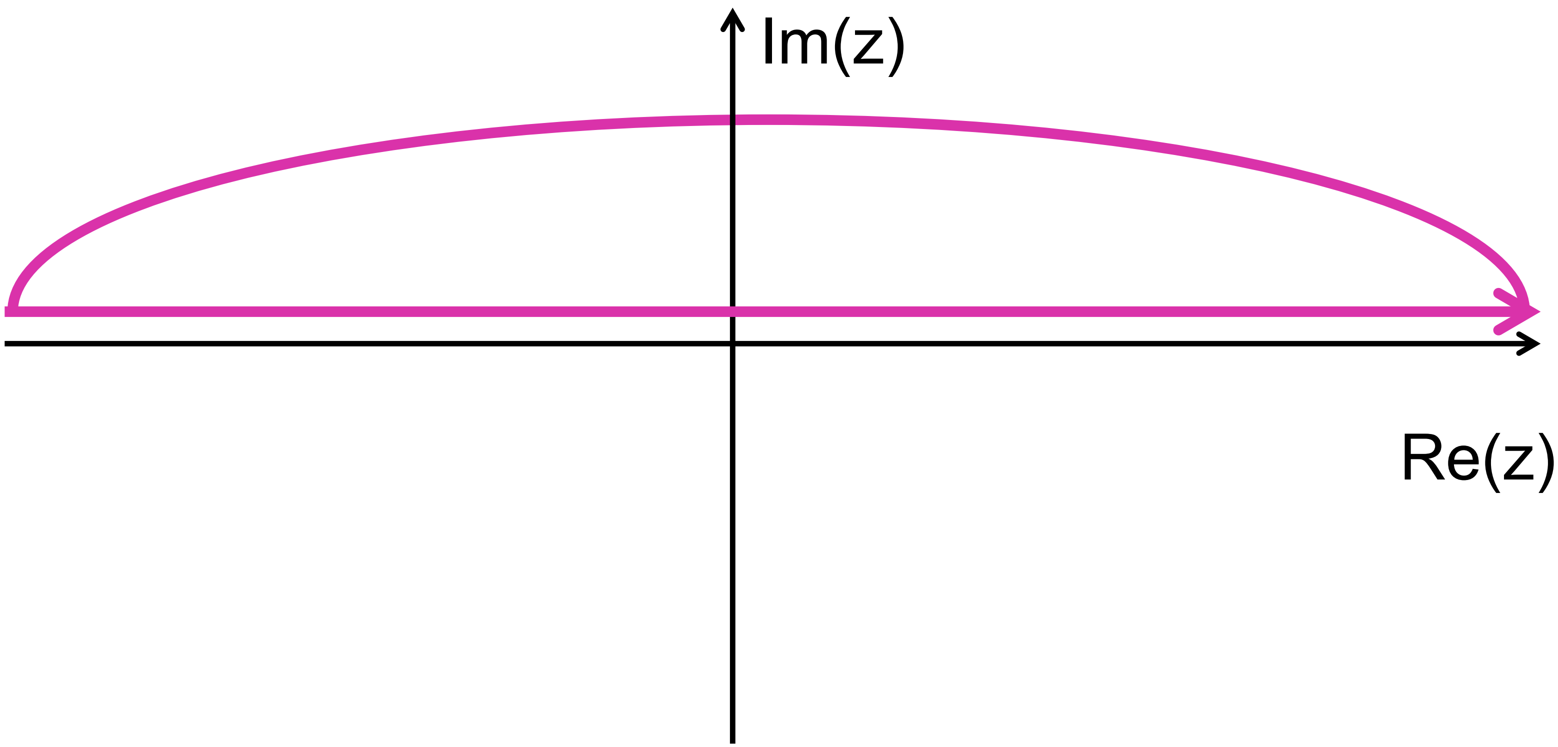
$$\text{Suppose } f\left(|z| \rightarrow \infty\right) = 0 \text{ and for } z = x:$$

$$f(x) = a(x) + ib(x)$$



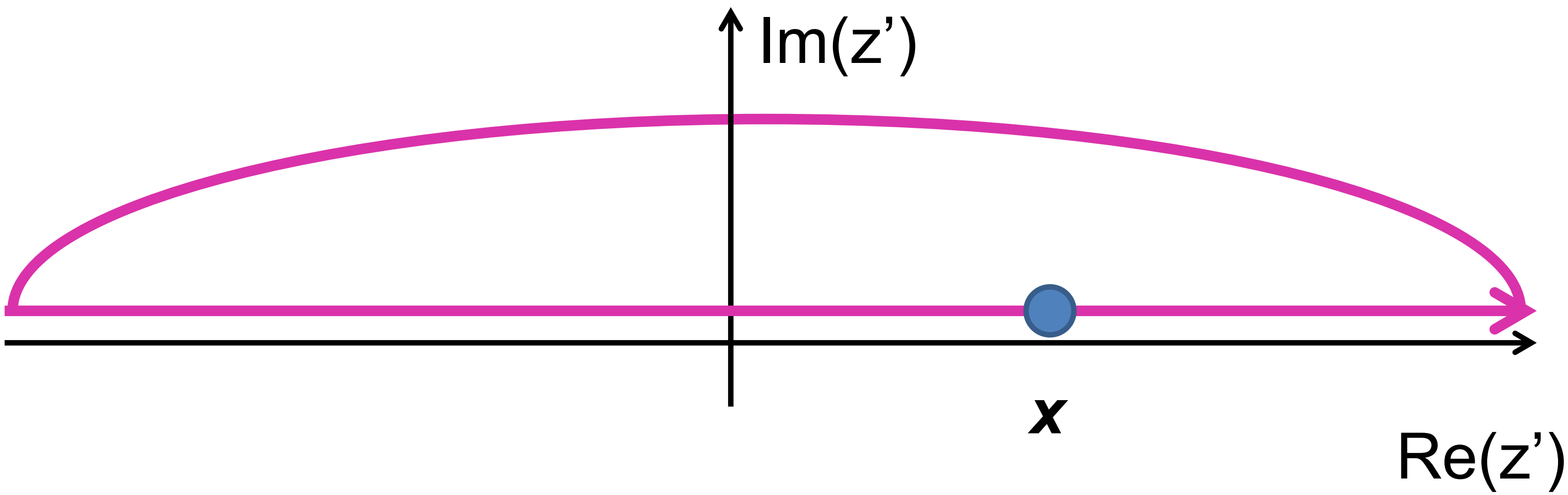
Example -- continued

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x)=a(x)+ib(x)$$



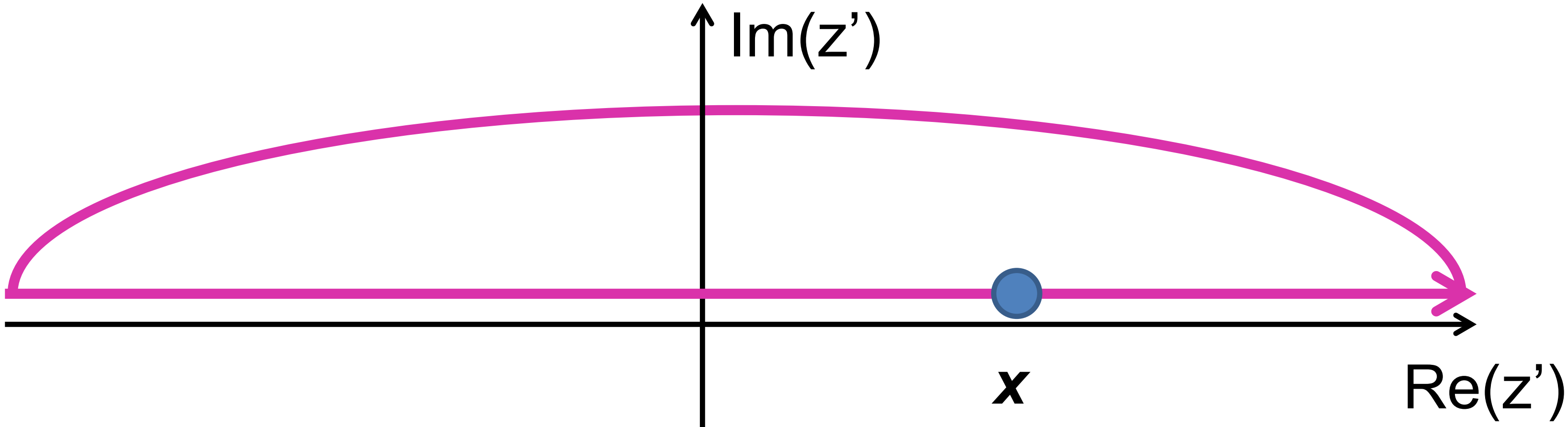
$$a(x)+ib(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+0$$

Example -- continued



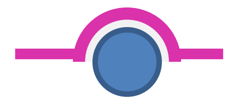
$$\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' = \int_{-\infty}^{x-\varepsilon} \frac{f(x')}{x'-x} dx' + \int_{x+\varepsilon}^{\infty} \frac{f(x')}{x'-x} dx' + \int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx'$$

$$= P \int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' + i\pi\, f(x)$$



let $u = x' - x$

let $x \to x + i\eta$

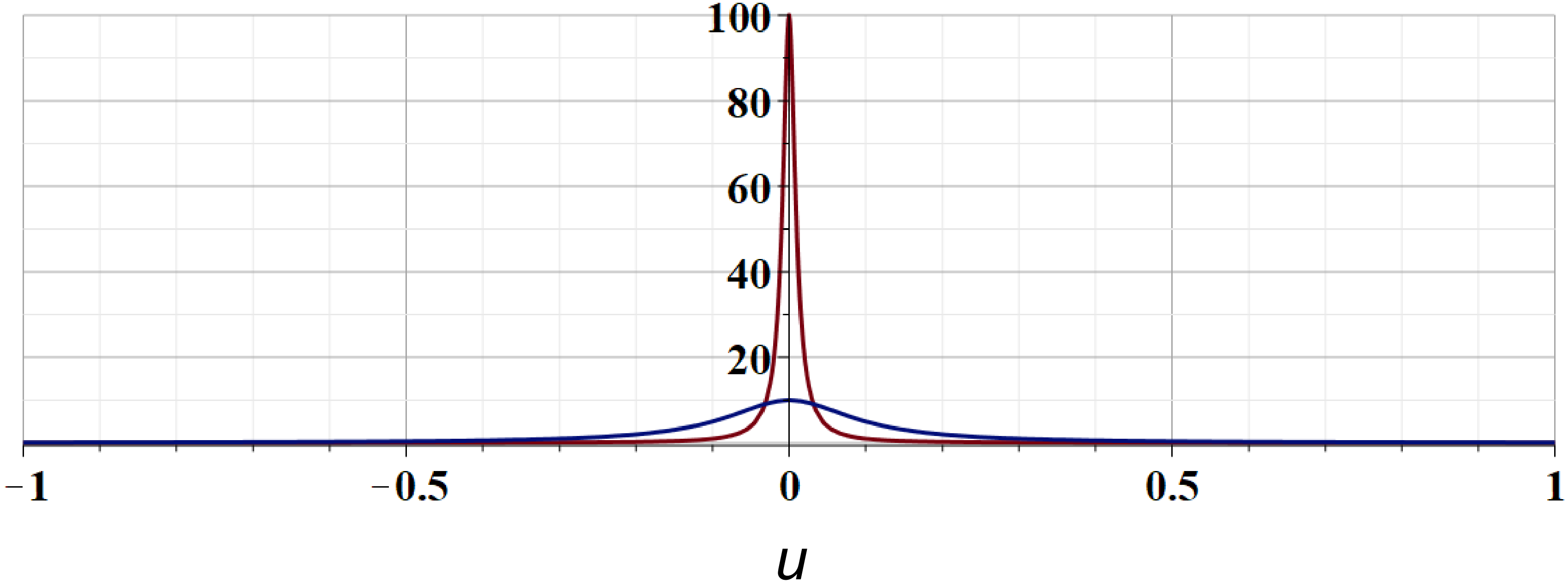


$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{f\left(x'\right)}{x'-x} dx' \approx f(x) \lim_{\eta \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{1}{u - i\eta} du = f(x) \lim_{\eta \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{u + i\eta}{u^2 + \eta^2} du$$

$$= i\pi f\left(x\right) \qquad \text{since} \qquad \lim_{\eta \to 0} \frac{i\eta}{u^2 + \eta^2} \approx i\pi\delta(u)$$

More details --

$$\lim_{\eta \to 0} \frac{\eta}{u^2 + \eta^2} \approx \pi\delta(u)$$



Example -- continued

$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' = \int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx' + \int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx' + \int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$= P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' + i\pi\, f(x)$$

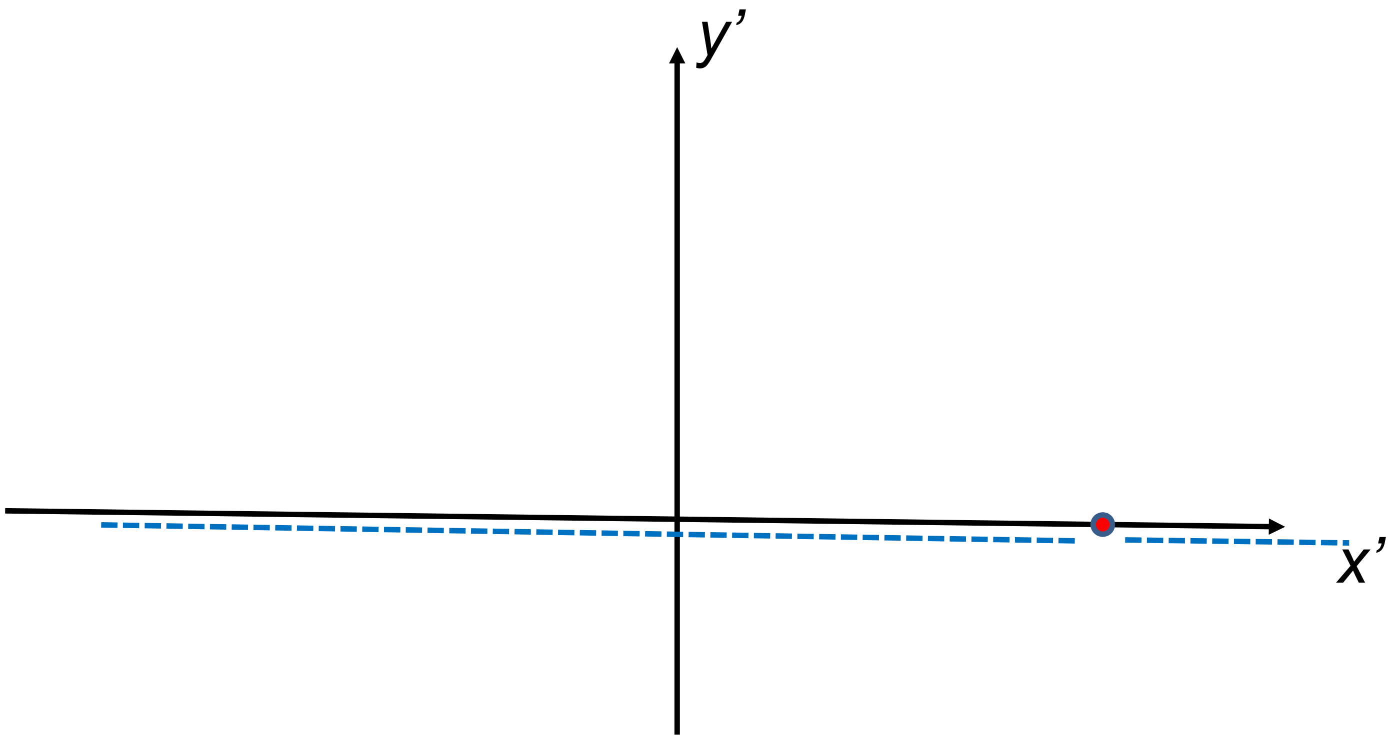
$$a(x)+ib(x) = \frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx' + \frac{\pi i}{2\pi i}\left(a(x)+ib(x)\right)$$

$$\Rightarrow a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

Kramers-Kronig relationships

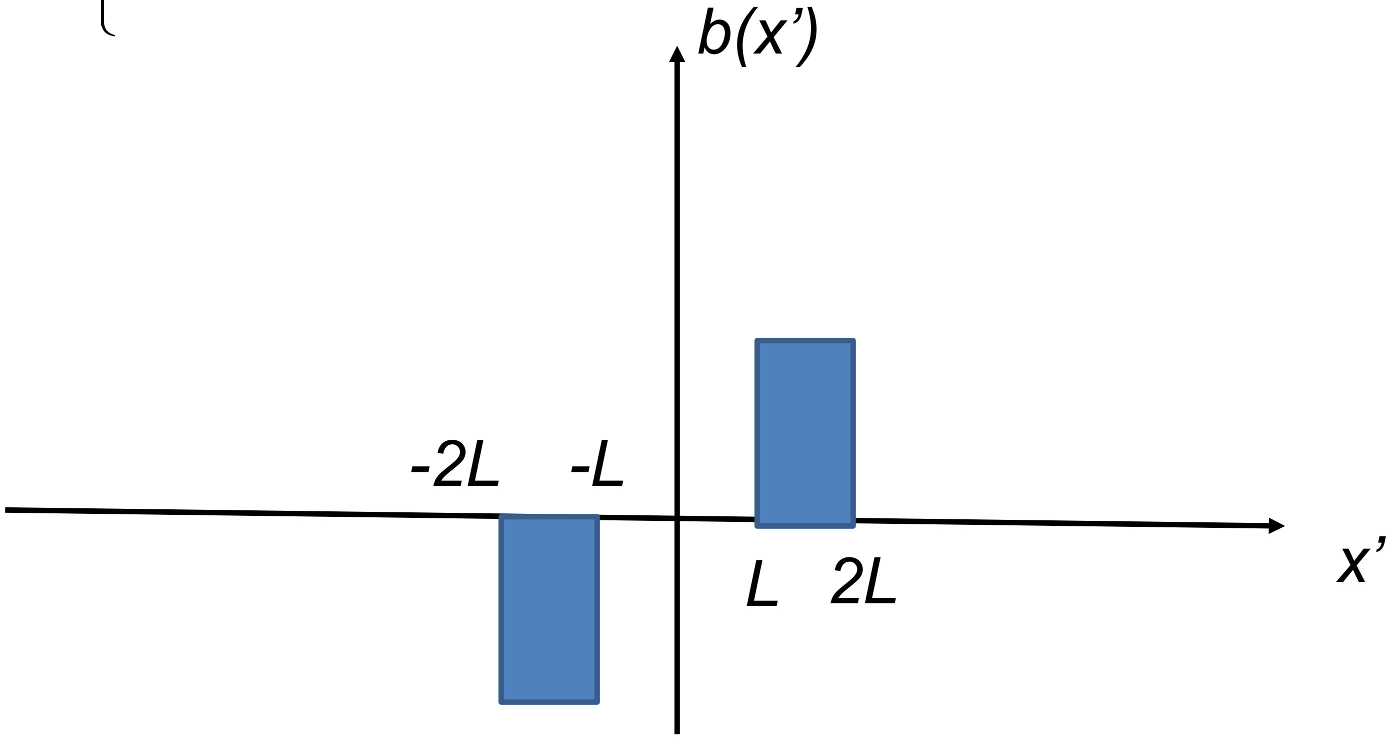
Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{b(x')}{x' - x} dx' = \lim_{\epsilon \to 0} \left(\frac{1}{\pi} \int_{-\infty}^{x-\epsilon} \frac{b(x')}{x' - x} dx' + \frac{1}{\pi} \int_{x+\epsilon}^{\infty} \frac{b(x')}{x' - x} dx' \right)$$



Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \quad L < x' < 2L \\ -B_0 & \text{for } \quad -2L < x' < -L \end{cases}$$



Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \quad L < x' < 2L \\ -B_0 & \text{for } \quad -2L < x' < -L \end{cases}$$

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x} dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x} dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x} dx' \right)$$

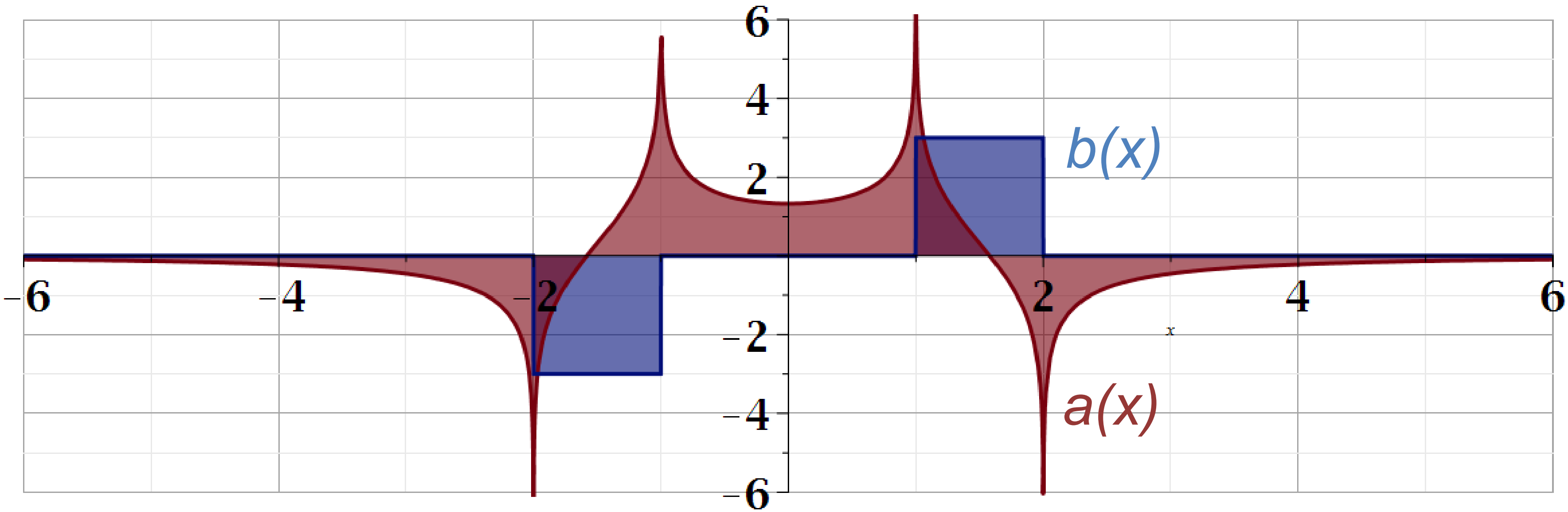
For $x < -2L \quad$ or $\quad x > 2L \qquad -L < x < L$:

$$a(x) = \frac{-B_0}{\pi}\int_{-2L}^{-L} \frac{dx'}{x'-x} + \frac{B_0}{\pi}\int_{L}^{2L} \frac{dx'}{x'-x}$$

$$= \frac{-B_0}{\pi}\ln\left(\left|\frac{x+L}{x+2L}\right|\right) + \frac{B_0}{\pi}\ln\left(\left|\frac{x-2L}{x-L}\right|\right) = \frac{B_0}{\pi}\ln\left(\left|\frac{x^2-4L^2}{x^2-L^2}\right|\right)$$

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$

For our example: $$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$



Summary

For a function $f(x)$, analytic along the real line:

$$f(x) = \Re\left(f(x)\right) + i\Im\left(f(x)\right) = a\left(x\right) + ib\left(x\right)$$

$$\Rightarrow a\left(x\right) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' \qquad b\left(x\right) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a\left(x'\right)}{x'-x}dx'$$

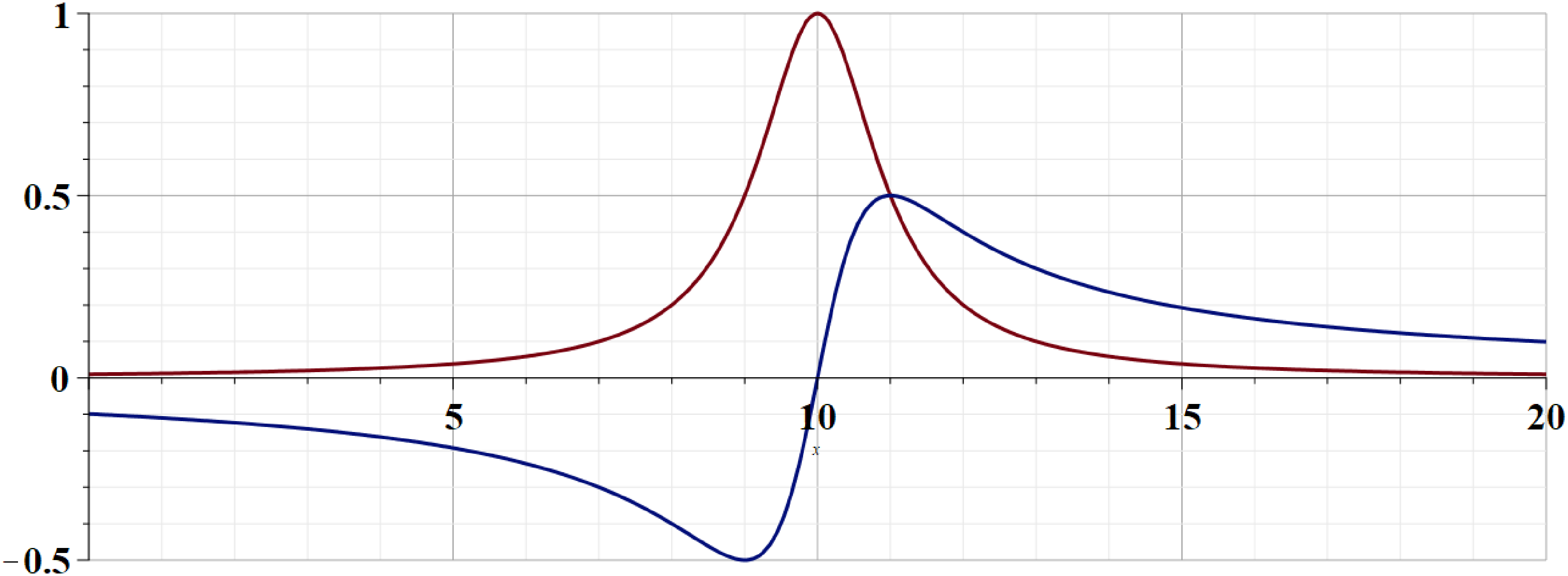
Example:

$$f(x) = \frac{1}{x+i} \qquad a(x) = \frac{x}{x^2+1} \qquad b(x) = -\frac{1}{x^2+1}$$

Check:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(x'-x\right)\left(x'^2+1\right)}dx' \overset{?}{=} \frac{x}{x^2+1} = a(x)$$

$$a(\omega) = \frac{\omega - 10}{(\omega - 10)^2 + 1} \qquad b(\omega) = \frac{1}{(\omega - 10)^2 + 1}$$



Continued:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x'-x)(x'^2+1)}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(x'-x)(x'^2+1)} - \frac{1}{(x'-x)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x^2-x'^2}{(x'-x)(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= \frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x+x'}{(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

Note that: $$\int_{x+\epsilon}^{X}\frac{1}{x'-x}dx' = \ln(X-x) - \ln(\epsilon) = \ln\left(\frac{X-x}{\epsilon}\right)$$

$$\int_{-X}^{x-\epsilon}\frac{1}{x'-x}dx' = -\ln(-X-x) + \ln(-\epsilon) = -\ln\left(\frac{X+x}{\epsilon}\right)$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx' = \lim_{X\to\infty}\ln\left(\frac{X-x}{X+x}\right) = 0 \qquad \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'^2+1}dx' = 1$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \frac{x}{x^2+1} = a(x)$$