# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF online or (occasionally in Olin 103

## Plan for Lecture 22: Chap. 7 & App. A-D (F&W)

**Generalization of the one dimensional wave equation → various mathematical problems and techniques including:**

1. **Fourier transforms**
2. **Laplace transforms**
3. **Complex variables**
4. **Contour integrals**

10/14/2020 PHY 711 Fall 2020 -- Lecture 22 1

In this lecture, we will focus on the mathematics of complex variables.

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| 1 | Wed, 8/26/2020 | Chap. 1 | Introduction | #1 | 8/31/2020 |
| 2 | Fri, 8/28/2020 | Chap. 1 | Scattering theory | #2 | 9/02/2020 |
| 3 | Mon, 8/31/2020 | Chap. 1 | Scattering theory | #3 | 9/04/2020 |
| 4 | Wed, 9/02/2020 | Chap. 1 | Scattering theory | | |
| 5 | Fri, 9/04/2020 | Chap. 1 | Scattering theory | #4 | 9/09/2020 |
| 6 | Mon, 9/07/2020 | Chap. 2 | Non-inertial coordinate systems | | |
| 7 | Wed, 9/09/2020 | Chap. 3 | Calculus of Variation | #5 | 9/11/2020 |
| 8 | Fri, 9/11/2020 | Chap. 3 | Calculus of Variation | #6 | 9/14/2020 |
| 9 | Mon, 9/14/2020 | Chap. 3 & 6 | Lagrangian Mechanics | #7 | 9/18/2020 |
| 10 | Wed, 9/16/2020 | Chap. 3 & 6 | Lagrangian & constraints | #8 | 9/21/2020 |
| 11 | Fri, 9/18/2020 | Chap. 3 & 6 | Constants of the motion | | |
| 12 | Mon, 9/21/2020 | Chap. 3 & 6 | Hamiltonian equations of motion | #9 | 9/23/2020 |
| 13 | Wed, 9/23/2020 | Chap. 3 & 6 | Liouville theorm | #10 | 9/25/2020 |
| 14 | Fri, 9/25/2020 | Chap. 3 & 6 | Canonical transformations | | |
| 15 | Mon, 9/28/2020 | Chap. 4 | Small oscillations about equilibrium | #11 | 10/02/2020 |
| 16 | Wed, 9/30/2020 | Chap. 4 | Normal modes of vibration | #12 | 10/05/2020 |
| 17 | Fri, 10/02/2020 | Chap. 4 | Normal modes of vibration | | |
| 18 | Mon, 10/05/2020 | Chap. 7 | Motion of strings | #13 | 10/07/2020 |
| 19 | Wed, 10/07/2020 | Chap. 7 | Sturm-Liouville equations | #14 | 10/09/2020 |
| 20 | Fri, 10/09/2020 | Chap. 7 | Sturm-Liouville equations | | |
| 21 | Mon, 10/12/2020 | Chap. 7 | Fourier transforms and Laplace transforms | | |
| 22 | Wed, 10/14/2020 | Chap. 7 | Complex variables and contour integration | | |



No new homework while working on mid term exam.

https://www.physics.wfu.edu/wfu-phy-news/seminars-2020-fall/

Online Colloquium: "Transition Metal Dichalcogenides: An Overview of Their Synthesis, Properties and Future Applications" — October 15, 2020 at 4 PM

Gabriel Marcus
Graduate Student
Mentor, Dr. David Carroll
Physics Department
Center for Nanotechnology and Molecular Materials
Wake Forest University, Winston-Salem, NC
Thursday, October 15, 2020 at 4:00 PM
Via Video Conference (contact wfuphys@wfu.edu for link information)

https://www.sciencedirect.com/science/article/pii/S2211285516301458 (Metallic 1T phase $MoS_2$ nanosheets for high-performance thermoelectric energy harvesting)

https://onlinelibrary.wiley.com/doi/full/10.1002/adma.201700070 (2D Chalcogenide Nanoplate Assemblies for Thermoelectric Applications)

https://onlinelibrary.wiley.com/doi/full/10.1002/adma.201702968 (Self-Assembled Heterostructures: Selective Growth of Metallic Nanoparticles on V2–VI3 Nanoplates)



Colloquium by senior graduate student who is working with Professor Carroll.

Introduction to complex variables

1. Basic properties
2. Notion of an analytic complex function
3. Cauchy integral theory
4. Evaluating integrals of functions in the complex plane



We will review/introduce the basic ideas associated with complex variables.

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\ e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$



First we consider the basic definitions and representations of a complex number. Then we consider a function of complex numbers. The Cauchy relationships follow from the notion that a function that is differentiable in the complex plane must have consistent partial derivatives along the real and imaginary axes.

## Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

## Examples of analytic functions

$$e^z = e^{x+iy} = e^x\cos(y) + ie^x\sin(y)$$

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x\sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = \left(x+iy\right)^2 = \left(x^2-y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$



Notion of an analytic function and an example that satisfies the conditions.

Examples of non-analytic functions

Note that $z = \rho e^{i\phi} = \rho e^{i\phi + i2\pi n}$ for any integer $n$

$\Rightarrow \ln z = \ln \rho + i(\phi + 2\pi n)$

$\ln z$ is not analytic because it is multivalued

$$\Rightarrow z^{\alpha} = \rho^{\alpha} e^{i\alpha\phi} e^{i2\pi n\alpha}$$

$z^{\alpha}$ is not analytic for non-integer $\alpha$ because it is multivalued

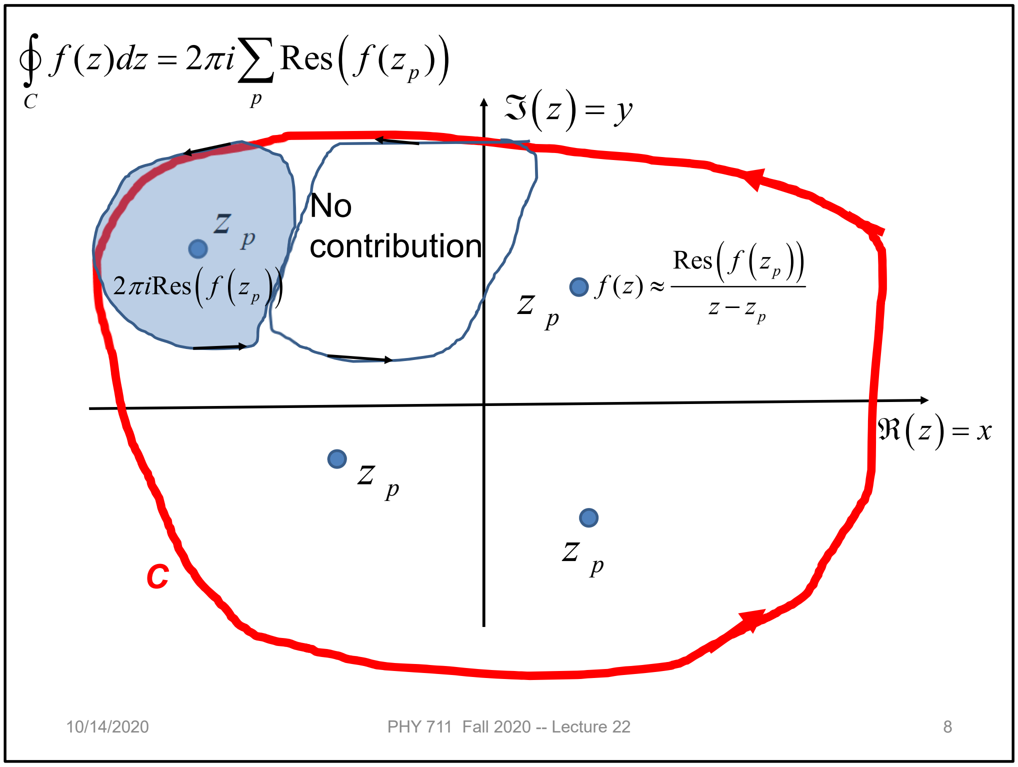
Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer $n$, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$



Examples of non-analytic functions. Special property of contour integrals about a function with a simple "pole".




Contributions to a closed contour from various contributions.

General formula for determining residue:

Suppose that in the neighborhood of $z_p$, $f(z) \approx \frac{g(z)}{\left(z - z_p\right)^m} \underset{z \to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z - z_p}$

Since $g(z)$ is analytic near $z_p$, we can make a Taylor exansion about $z_p$ :

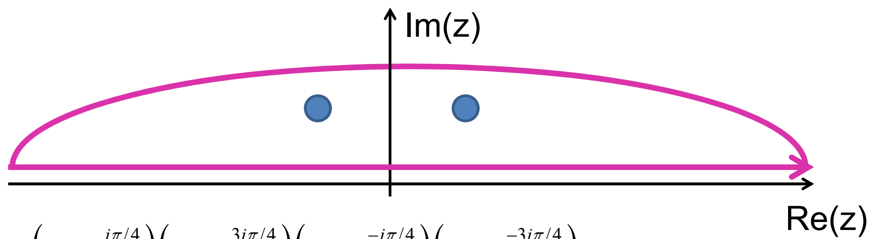
$$g(z) \approx g(z_p) + \left(z - z_p\right) \frac{dg(z_p)}{dz} + \ldots + \frac{\left(z - z_p\right)^{m-1}}{(m-1)!} \frac{d^{m-1} g(z_p)}{dz^{m-1}} + \ldots.$$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z \to z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}\left(\left(z - z_p\right)^m f(z)\right)}{dz^{m-1}} \right\}$$



Residue theorem

Example: $$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx=\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx+0=\oint\frac{z^2}{1+z^4}dz$$




$$1+z^4=\left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint\frac{z^2}{1+z^4}dz=2\pi i\left(\text{Res}\left(z_p=e^{i\pi/4}\right)+\text{Res}\left(z_p=e^{3i\pi/4}\right)\right)$$

Note: *m=1*

$$\text{Res}\left(z_p=e^{i\pi/4}\right)=\frac{e^{i\pi/4}}{4i}\qquad\text{Res}\left(z_p=e^{3i\pi/4}\right)=-\frac{e^{3i\pi/4}}{4i}$$

$$\oint\frac{z^2}{1+z^4}dz=2\pi i\left(\frac{e^{i\pi/4}}{4i}-\frac{e^{3i\pi/4}}{4i}\right)=\frac{\pi}{2}\left(\left(\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}\right)-\left(-\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}\right)\right)=\frac{\pi}{\sqrt{2}}$$



Example of a contour integral using the residue theorem.

Some details:

$$f(z) = \frac{z^2}{1+z^4}$$

$$\text{Res}\left(f(z = e^{i\pi/4}\right) = \frac{\left(e^{i\pi/4}\right)^2}{\left(e^{i\pi/4} - e^{3i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-3i\pi/4}\right)}$$

$$= \frac{e^{i\pi/2}}{\left(e^{i\pi/4} + e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} + e^{i\pi/4}\right)}$$

$$= \frac{e^{i\pi/4}}{2\left(i - (-i)\right)} = \frac{e^{i\pi/4}}{4i}$$



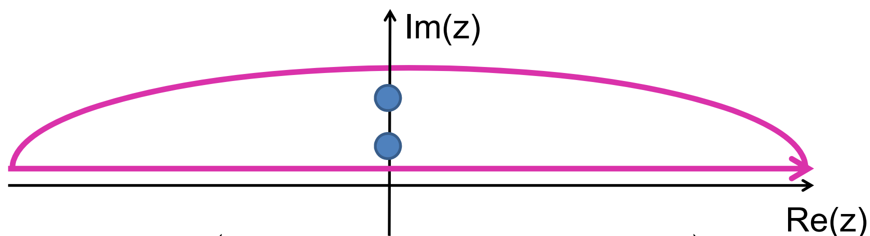
Some details.

Another example: $I = \int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx$.

$$\int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz$$

$$4z^4 + 5z^2 + 1 = 4(z-i)\left(z - \tfrac{i}{2}\right)(z+i)\left(z + \tfrac{i}{2}\right)$$

Note: *m=1*




$$I = 2\pi i\left(\mathrm{Res}\left(z_p = i\right) + \mathrm{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

Another example.

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$= 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$= \frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

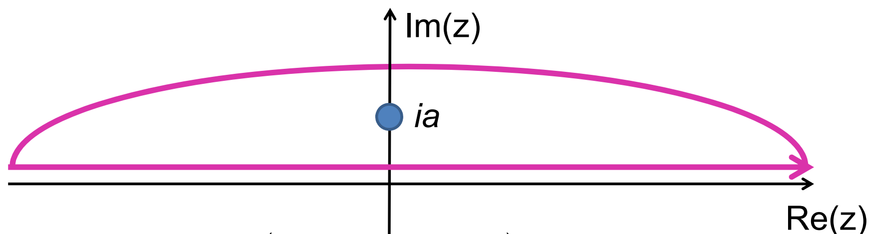


Some details.

Another example: $I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx \quad \text{for } k > 0 \text{ and } a > 0$

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$




$$I = 2\pi i \left( \text{Res}\left( z_p = ia \right) \right) = \pi e^{-ka}$$



Another example

Some details --

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$

$$\frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz = 2\pi i \frac{1}{i} \lim_{z \to ia} \left( (z - ia) \frac{z e^{ikz}}{z^2 + a^2} \right)$$

$$= 2\pi i \frac{1}{i} \frac{ia e^{-ka}}{2ia} = \pi e^{-ka}$$



More details.

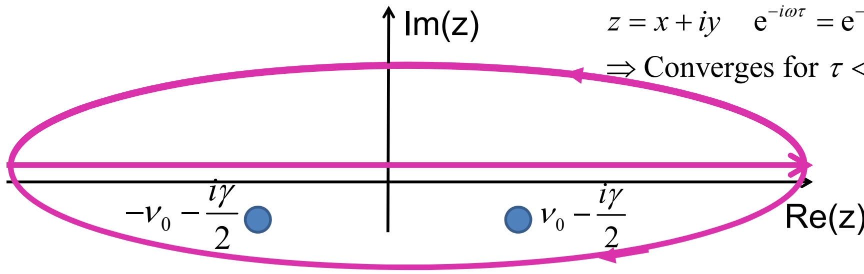
From the Drude model of dielectric response --

$$G(\tau)=\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}$$ where $\omega_p$, $\omega_0$, and $\gamma$ are positive constants

Upper hemisphere:

$z=x+iy \quad e^{-i\omega\tau}=e^{-ix\tau+y\tau}$

$\Rightarrow$ Converges for $\tau<0$




Lower hemisphere:

$z=x-iy \quad e^{-i\omega\tau}=e^{-ix\tau-y\tau}$

$\Rightarrow$ Converges for $\tau>0$

$$\nu_0\equiv\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$$

From the Drude model of dielectric response -- continued --

$$G(\tau) = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$ where $\omega_p$, $\omega_0$, and $\gamma$ are positive constants

$$G(\tau) = \omega_p^2 \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-\gamma\tau/2} \dfrac{\sin \nu_0 \tau}{\nu_0} & \text{for } \tau > 0 \end{cases}$$



Another example from the Drude model.

Cauchy integral theorem for analytic function *f(z)*:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'.$$

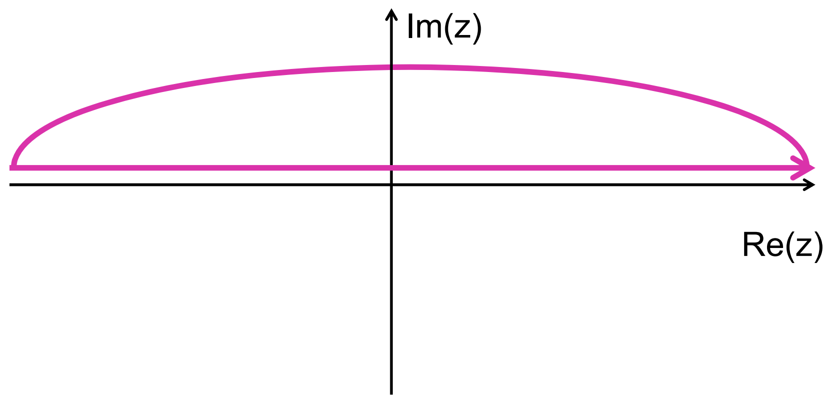


Another useful theorem from Cauchy.

Example

Suppose $f\left(|z| \rightarrow \infty\right) = 0$ and for $z = x$:

$$f(x) = a(x) + ib(x)$$

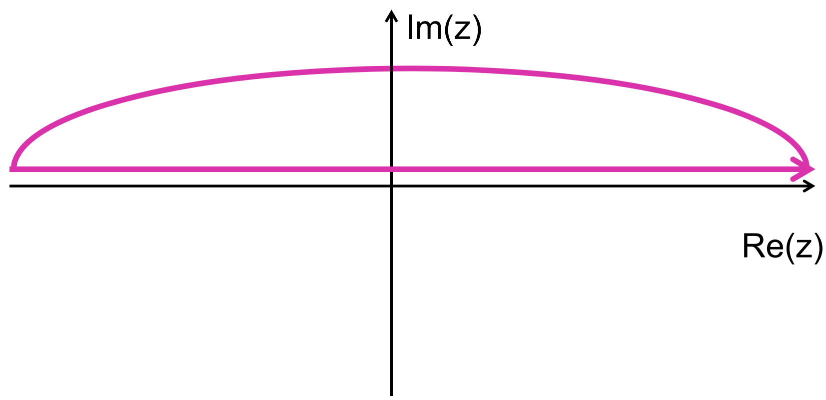





Considering real and imaginary parts.

Example -- continued

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x)=a(x)+ib(x)$$

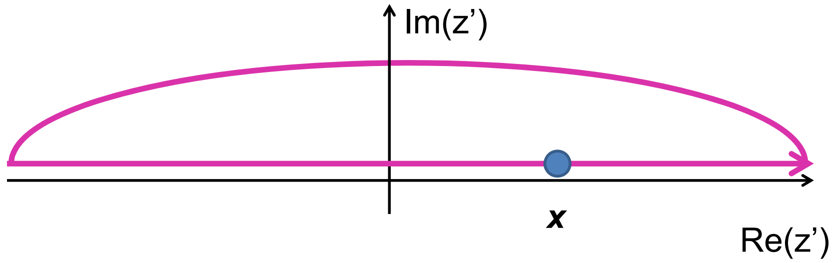



$$a(x)+ib(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+0$$



Real and imaginary parts for Cauchy’s integral relation.

Example -- continued

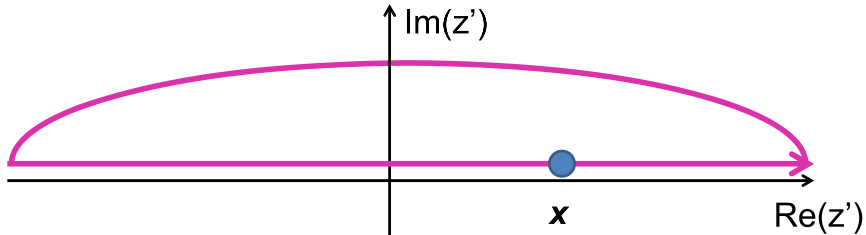



$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' = \int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx' + \int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx' + \int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$= P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' + i\pi\, f(x)$$

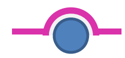


Detail of how to evaluate the point x=x' in terms of the principal parts integral.




let $u = x' - x$

let $x \to x + i\eta$



$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x}\,dx' \approx f(x)\lim_{\eta\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{1}{u-i\eta}\,du = f(x)\lim_{\eta\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{u+i\eta}{u^2+\eta^2}\,du$$

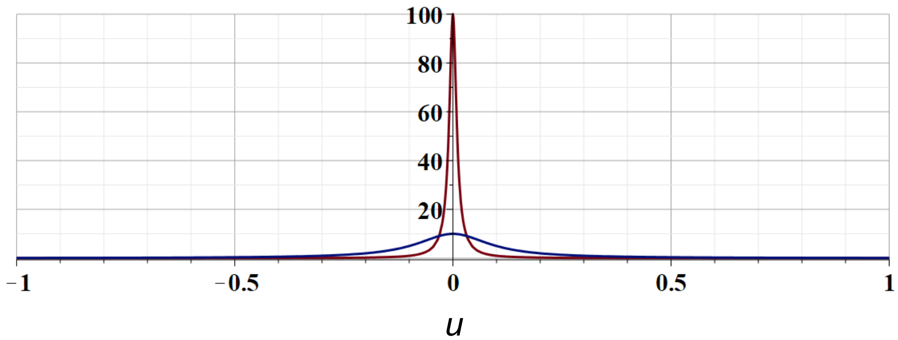
$$= i\pi f(x) \qquad \text{since} \quad \lim_{\eta\to 0}\frac{i\eta}{u^2+\eta^2} \approx i\pi\delta(u)$$



Some details.

More details --

$$\lim_{\eta \to 0} \frac{\eta}{u^2 + \eta^2} \approx \pi\delta(u)$$






Justification of the delta function result.

Example -- continued

$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' = \int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx' + \int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx' + \int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$= P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' + i\pi\, f(x)$$

$$a(x)+ib(x) = \frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx' + \frac{\pi i}{2\pi i}\left(a(x)+ib(x)\right)$$

$$\Rightarrow a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

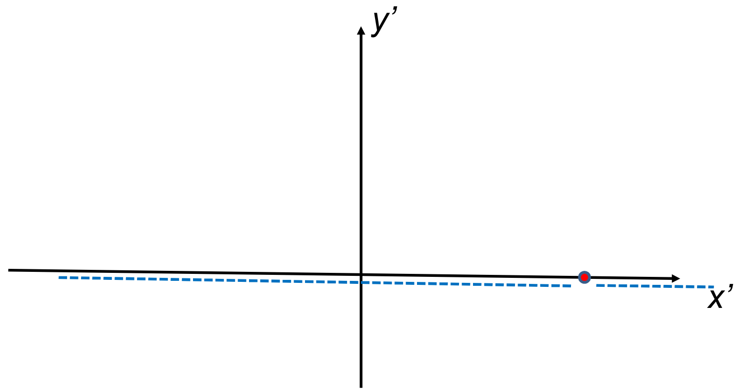
Kramers-Kronig relationships



Final result relating real and imaginary parts of complex function.

## Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx'\right)$$

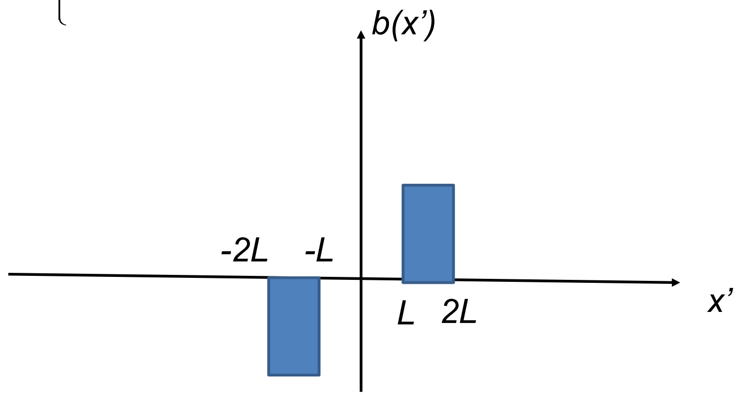





Some details.

Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \; L < x' < 2L \\ -B_0 & \text{for } \; -2L < x' < -L \end{cases}$$






Simple example for complex function.

Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \; L < x' < 2L \\ -B_0 & \text{for } \; -2L < x' < -L \end{cases}$$

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left( \frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx' \right)$$

For $x < -2L \quad \text{or} \quad x > 2L \quad -L < x < L$:

$$a(x) = \frac{-B_0}{\pi}\int_{-2L}^{-L} \frac{dx'}{x'-x} + \frac{B_0}{\pi}\int_{L}^{2L} \frac{dx'}{x'-x}$$

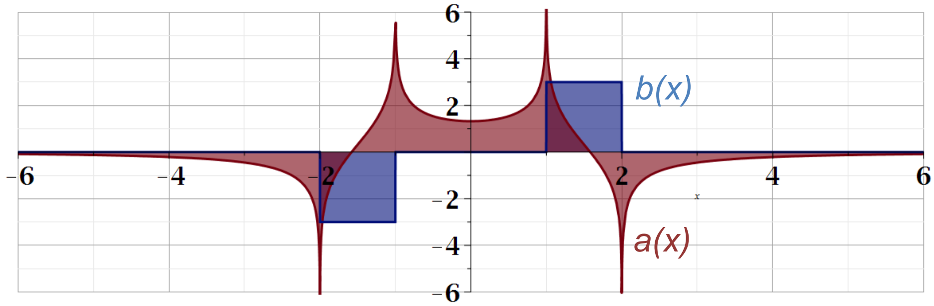
$$= \frac{-B_0}{\pi}\ln\left( \left| \frac{x+L}{x+2L} \right| \right) + \frac{B_0}{\pi}\ln\left( \left| \frac{x-2L}{x-L} \right| \right) = \frac{B_0}{\pi}\ln\left( \left| \frac{x^2-4L^2}{x^2-L^2} \right| \right)$$



For given imaginary function, this is the form of the real function.

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx'\right)$$

For our example: $$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$






Summary of results.

## Summary

For a function $f(x)$, analytic along the real line:

$$f(x) = \Re\left(f(x)\right) + i\Im\left(f(x)\right) = a\left(x\right) + ib\left(x\right)$$

$$\Rightarrow a\left(x\right) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' \qquad b\left(x\right) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a\left(x'\right)}{x'-x}dx'$$

Example:

$$f(x) = \frac{1}{x+i} \qquad a(x) = \frac{x}{x^2+1} \qquad b(x) = -\frac{1}{x^2+1}$$

Check:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(x'-x\right)\left(x'^2+1\right)}dx' \stackrel{?}{=} \frac{x}{x^2+1} = a(x)$$



Summary.

Continued:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x'-x)(x'^2+1)}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(x'-x)(x'^2+1)} - \frac{1}{(x'-x)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x^2-x'^2}{(x'-x)(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= \frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x+x'}{(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

Note that: $\int_{x+\epsilon}^{X}\frac{1}{x'-x}dx' = \ln(X-x) - \ln(\epsilon) = \ln\left(\frac{X-x}{\epsilon}\right)$

$$\int_{-X}^{x-\epsilon}\frac{1}{x'-x}dx' = -\ln(-X-x) + \ln(-\epsilon) = -\ln\left(\frac{X+x}{\epsilon}\right)$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx' = \lim_{X\to\infty}\ln\left(\frac{X-x}{X+x}\right) = 0 \qquad \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'^2+1}dx' = 1$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \frac{x}{x^2+1} = a(x)$$