

PHY 711 Classical Mechanics and Mathematical Methods

**10-10:50 AM MWF online or (occasionally in
Olin 103**

Discussion for Lecture 24 – Chap. 5 (F &W)

Rotational motion

- 1. Torque free motion of a rigid body**
- 2. Rigid body motion in body fixed frame**
- 3. Conversion between body and inertial reference frames**
- 4. Symmetric top motion**

Reminder – take-home due at 5 PM this evening

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)

Tim – 9 AM Tuesday

Gao – 9 PM Tuesday

Jeanette – 11 AM Wednesday

Derek – 12 PM Friday

Your questions –

From Tim –

1. So for the problem on slide 23, if there is no initial movement/rotation of the top then the effective potential would stay as $Mgl\cos(\beta)$.

From Nick –

1. Can you explain the green line N in the diagram? I looked it up and it's called the line of nodes.
2. Are the Euler angles a representation of how the basis axes in the inertial frame are rotated? And what is the direction of the rotation?

From Gao –

1. How to decide stable/unstable solutions in slide 25?

16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
 24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 5	Rigid body motion	#16	10/23/2020

Due Wednesday

10/19/2020

PHY 711 – Homework # 15

Read Appendix A of **Fetter and Walecka**.

1. Assume that $a > 0$ and $b > 0$; use contour integration methods to evaluate the integral:

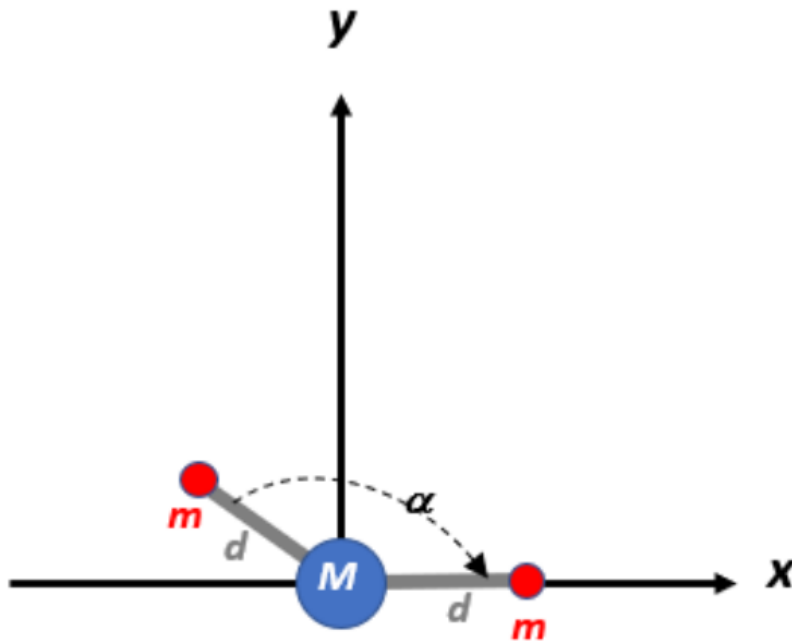
$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2} dx.$$

Note that you may use Maple, Mathematica, or other software to evaluate this integral, but full credit will be earned by using the contour integration methods.

PHY 711 – Homework # 16

Read Chap. 5 of Fetter and Walecka.

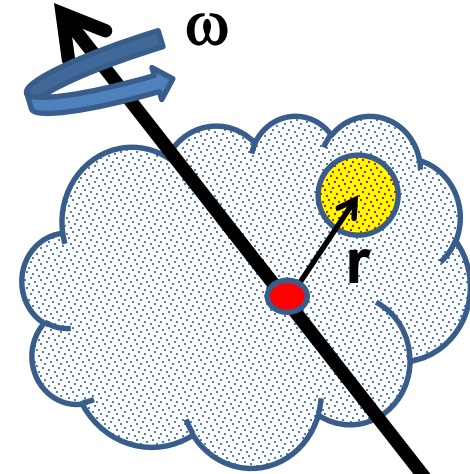
Due Friday



1. The figure above shows a rigid 3 atom molecule placed in the $x - y$ plane as shown.
 - (a) Find the moment of inertia tensor in the given coordinate system placed at the center of mass M in terms of the masses, bond lengths d and angle α .
 - (b) Find the principal moments I_1 , I_2 , and I_3 and the corresponding principal axes.
 - (c) (Extra credit) Find the principal moments and axes for a coordinate system centered at the center of mass of the molecule.

Summary of previous results
describing rigid bodies rotating
about a fixed origin ●

$$\left(\frac{d\mathbf{r}}{dt} \right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$



Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left| \boldsymbol{\omega} \times \mathbf{r}_p \right| \right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p \right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p \right)$$

$$= \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \vec{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \vec{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

Moment of inertia tensor

Matrix notation:

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

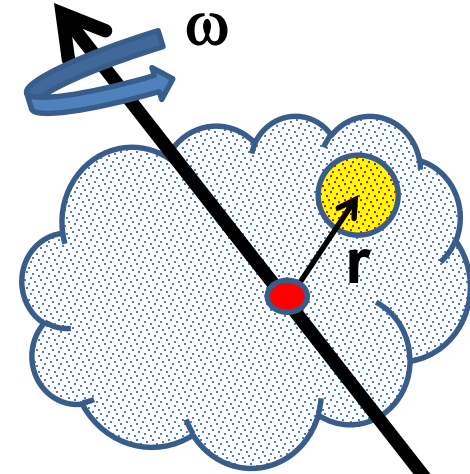
For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes

moment of inertia tensor: $\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \quad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt} \right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega} \times \mathbf{r}_p)$

$$\mathbf{L} = \sum_p m_p \left[\boldsymbol{\omega} (\mathbf{r}_p \cdot \mathbf{r}_p) - \mathbf{r}_p (\mathbf{r}_p \cdot \boldsymbol{\omega}) \right]$$

$$\mathbf{L} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \vec{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative:
$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = & I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \\ & \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 \end{aligned}$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3$$

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Want to determine
angular velocities $\omega_i(t)$

Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$

$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 = 0 \quad \Rightarrow \quad \tilde{\omega}_3 = (\text{constant})$$

Define : $\Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega$$

$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\text{Solution : } \tilde{\omega}_1(t) = A \cos(\Omega t + \varphi)$$

$$\tilde{\omega}_2(t) = A \sin(\Omega t + \varphi)$$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$= I_1 A (\cos(\Omega t + \varphi) \hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi) \hat{\mathbf{e}}_2) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

For example, the
object starts spinning
along the 3 axis.

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

Euler equations for asymmetric top -- continued

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1 \tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2 \tilde{\omega}_1$$

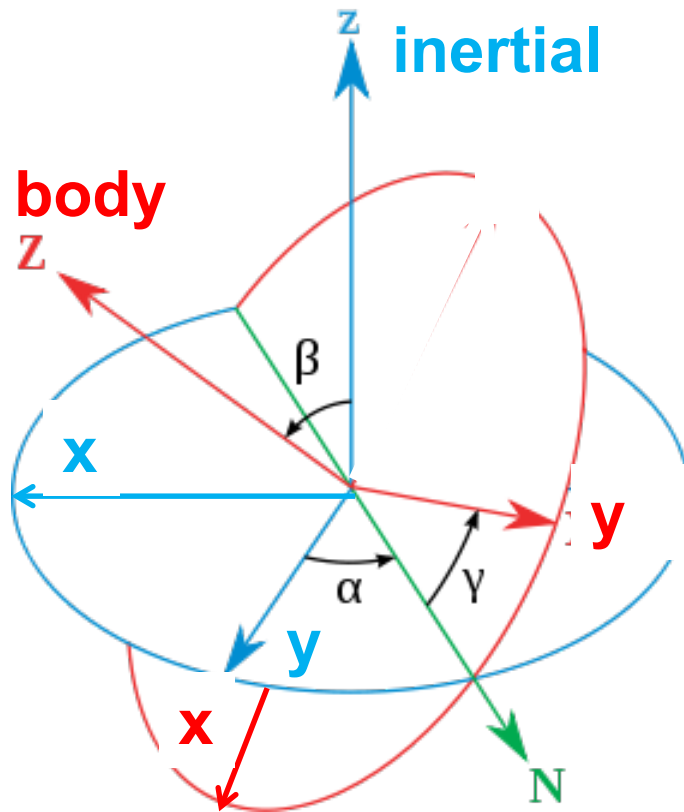
If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A \cos(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$$\tilde{\omega}_2(t) \approx A \sqrt{\frac{\Omega_2}{\Omega_1}} \sin(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

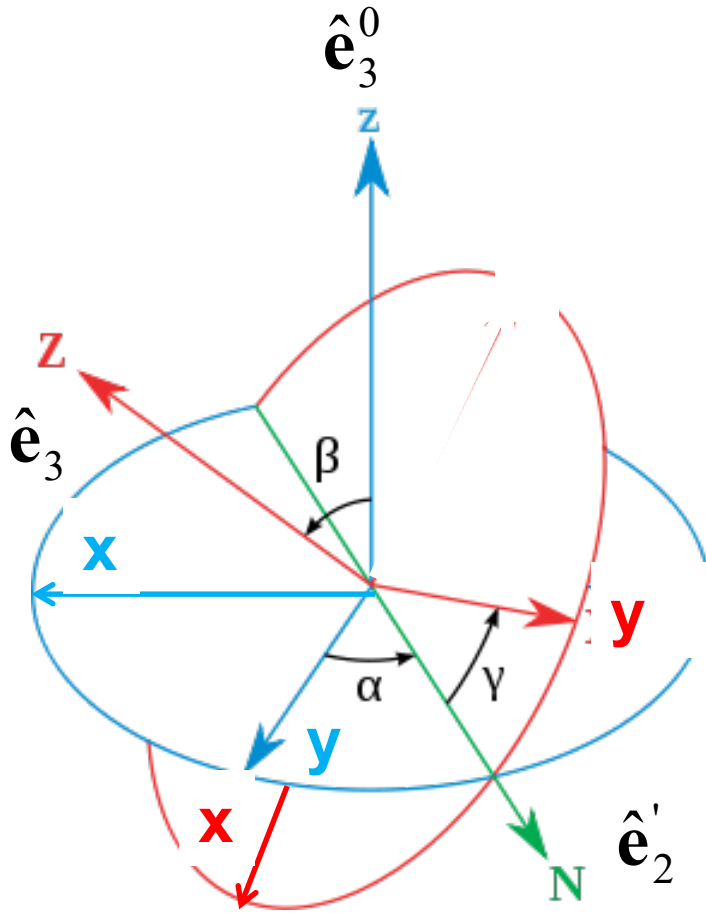
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

Your questions -- Can you explain the green line N in the diagram? I looked it up and it's called the line of nodes. Are the Euler angles a representation of how the basis axes in the inertial frame are rotated? And what is the direction of the rotation?

Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in most quantum mechanics texts and NOT the convention found in most classical mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial 3 axis and another is along the body fixed 3 axis. The middle rotation is along an intermediate 2 axis.

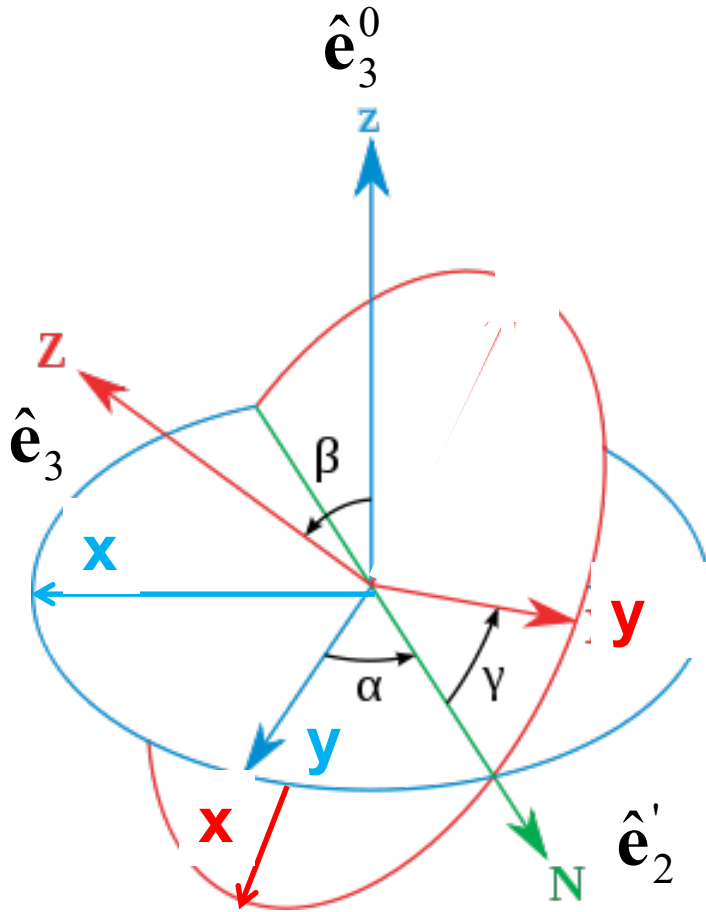


$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\omega} = \tilde{\omega}_1 \hat{e}_1 + \tilde{\omega}_2 \hat{e}_2 + \tilde{\omega}_3 \hat{e}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$

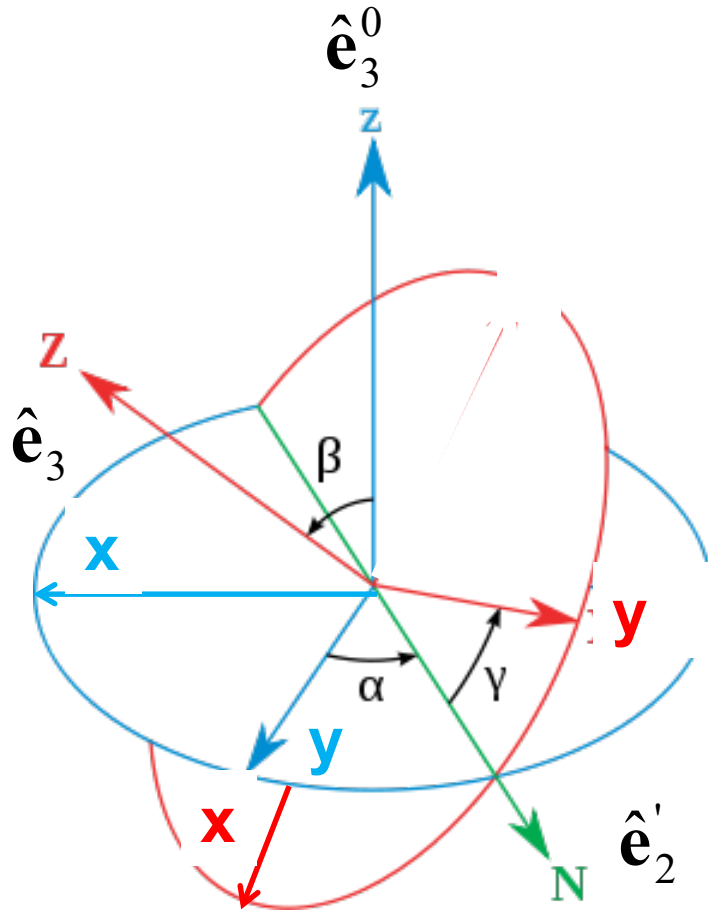


$$\hat{\mathbf{e}}_2' = \sin \gamma \hat{\mathbf{e}}_1 + \cos \gamma \hat{\mathbf{e}}_2$$

Matrix representation :

$$\hat{\mathbf{e}}_2' = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$



$$\begin{aligned} \hat{\mathbf{e}}_3^0 &= -\sin \beta \hat{\mathbf{e}}_1' + \cos \beta \hat{\mathbf{e}}_3' \\ &= -\cos \gamma \sin \beta \hat{\mathbf{e}}_1 + \sin \gamma \sin \beta \hat{\mathbf{e}}_2 + \cos \beta \hat{\mathbf{e}}_3 \end{aligned}$$

Matrix representation:

$$\begin{aligned} \hat{\mathbf{e}}_3^0 &= \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} \end{aligned}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

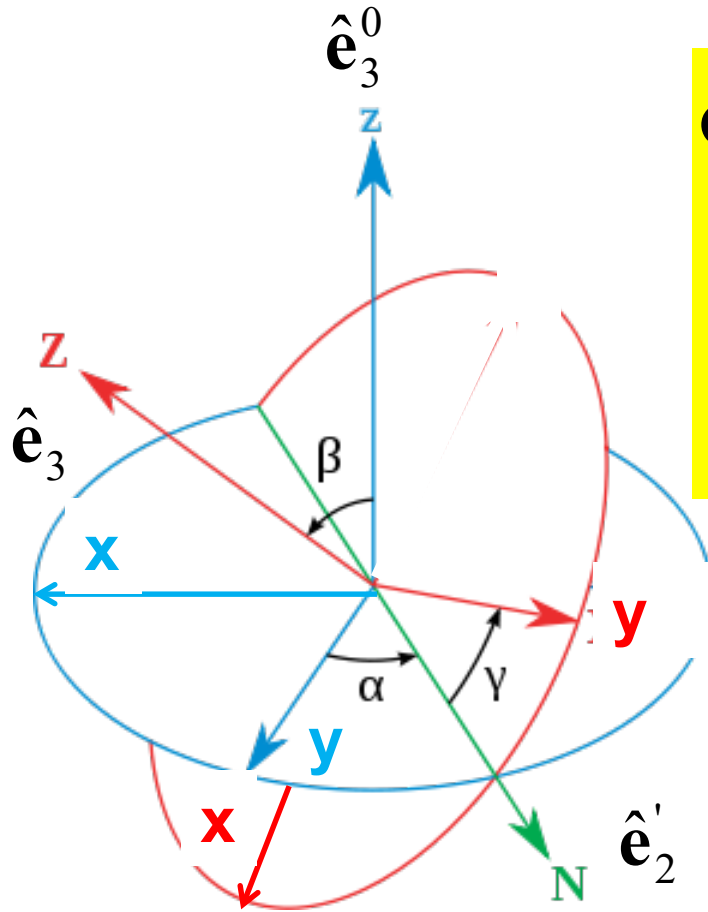
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma$$

$$\tilde{\omega}_3 = \dot{\alpha} \cos \beta + \dot{\gamma}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$



$$\begin{aligned} \tilde{\boldsymbol{\omega}} = & \left[\dot{\alpha} (-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma \right] \hat{\mathbf{e}}_1 \\ & + \left[\dot{\alpha} (\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma \right] \hat{\mathbf{e}}_2 \\ & + \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right] \hat{\mathbf{e}}_3 \end{aligned}$$

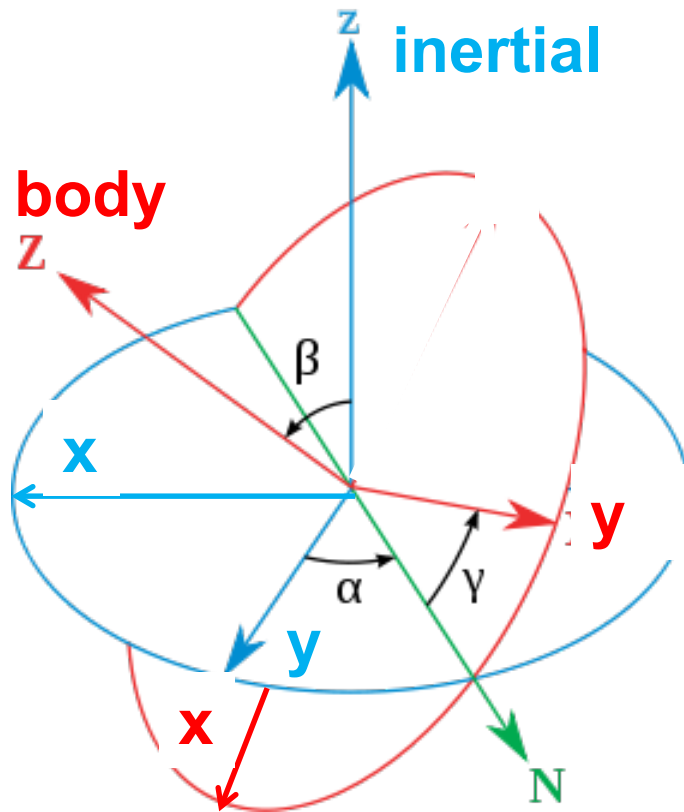
Rotational kinetic energy

$$\begin{aligned}T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\&= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2 \\&\quad + \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2 \\&\quad + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2\end{aligned}$$

If $I_1 = I_2$:

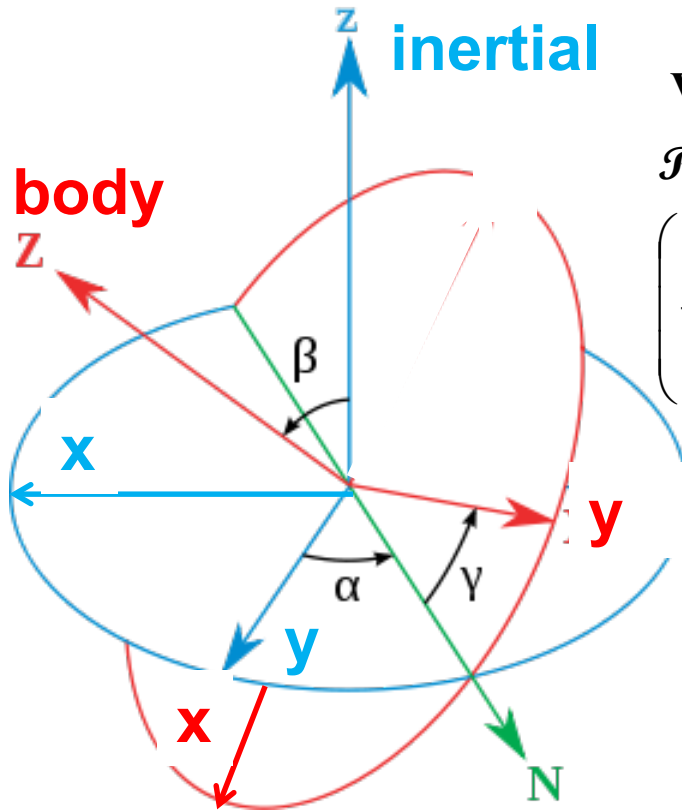
$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

General transformation between rotated coordinates – Euler angles



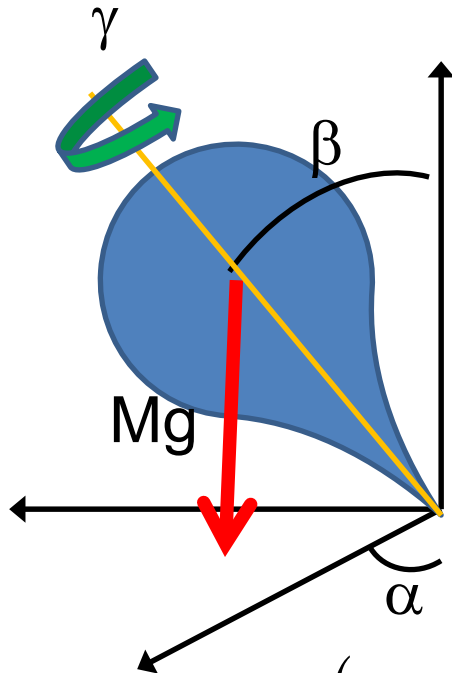
$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} =$$

$$\begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) +$$

$$\frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

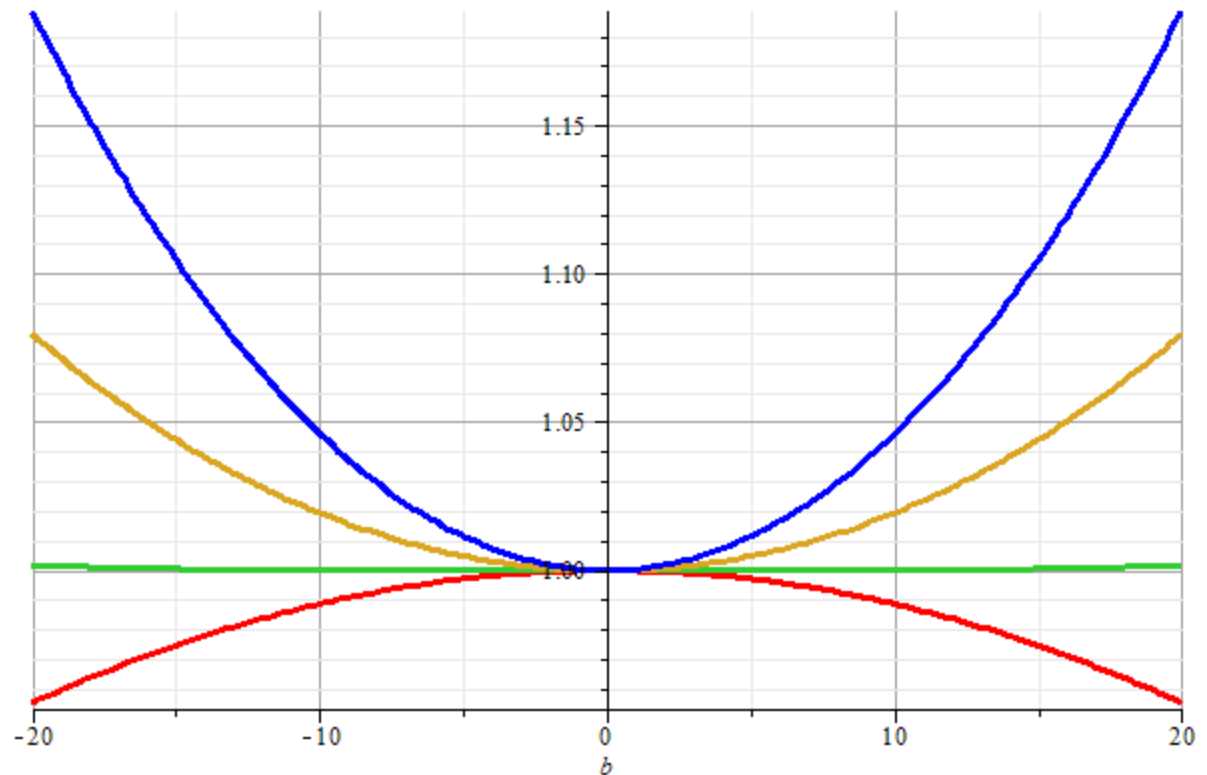
$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

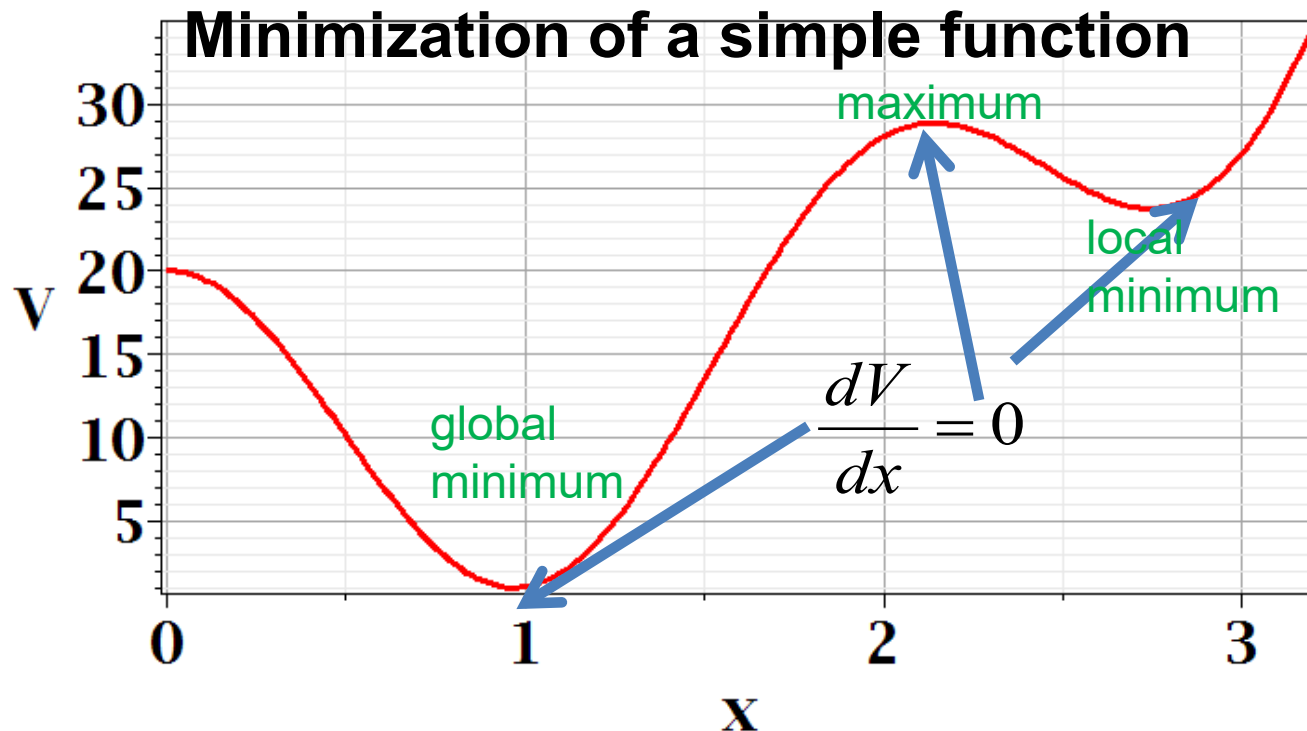
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Stable/unstable
solutions near
 $\beta=0$



Your questions --How to decide stable/unstable solutions in slide 25? So for the problem on slide 23, if there is no initial movement/rotation of the top then the effective potential would stay as $Mgl\cos(\beta)$.

Comment – When we discussed one dimensional motion, we discussed stable and unstable equilibrium points. At equilibrium $dV/dx=0$, but only when $V(x)$ has a minimum at that point, is the system stable in the sense that for small displacements from equilibrium, there are restoring forces to move the system back to the equilibrium point.



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_1} \frac{(1 - 1 + \frac{1}{2} \beta^2)^2}{\beta^2} + Mgl(1 - \frac{1}{2} \beta^2)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2} \right) \beta^2 + Mgl$$

$\Rightarrow$ Stable solution if

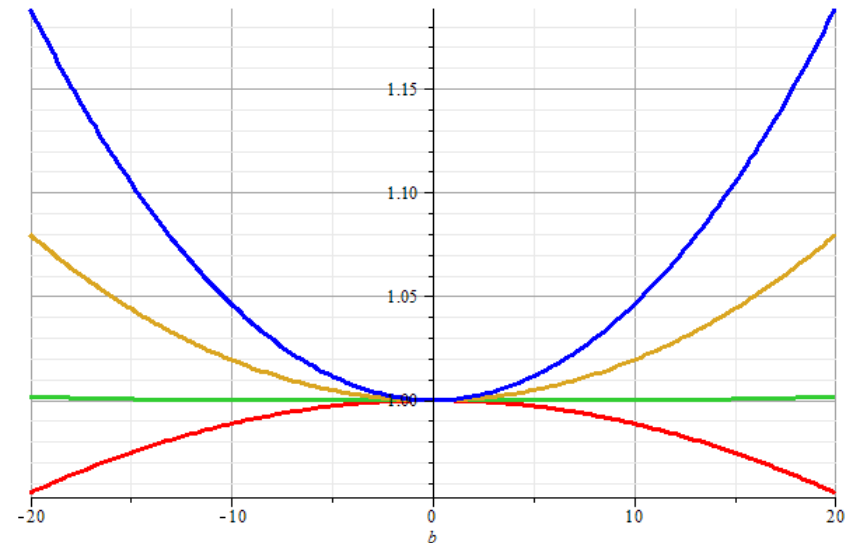
$$p_\gamma \geq \sqrt{4MglI_1}$$

Note that

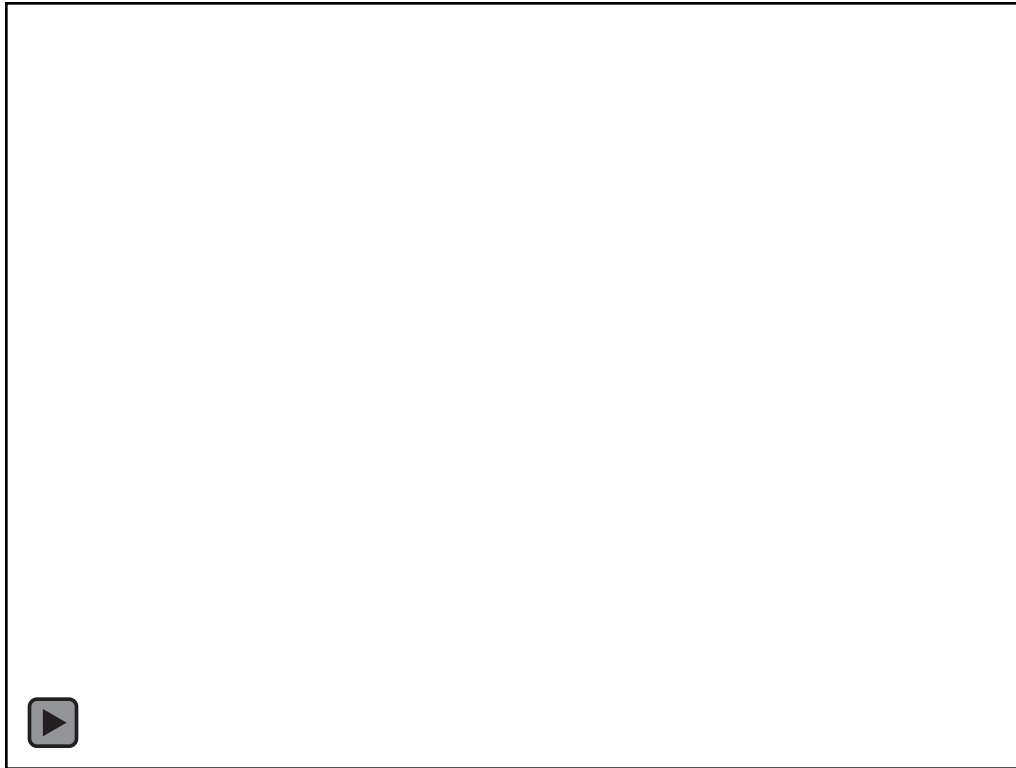
$$p_\gamma = I_3 \omega_3$$

$\Rightarrow \omega_3$ must be sufficiently large

for the top to maintain vertical orientation ($\beta \approx 0$).



<http://www.physics.usyd.edu.au/~cross/SPINNING%20TOPS.htm>



[Home](#) > [American Journal of Physics](#) > [Volume 81, Issue 4](#) > [10.1119/1.4776195](#)

 Full . Published Online: 18 March 2013 Accepted: December 2012

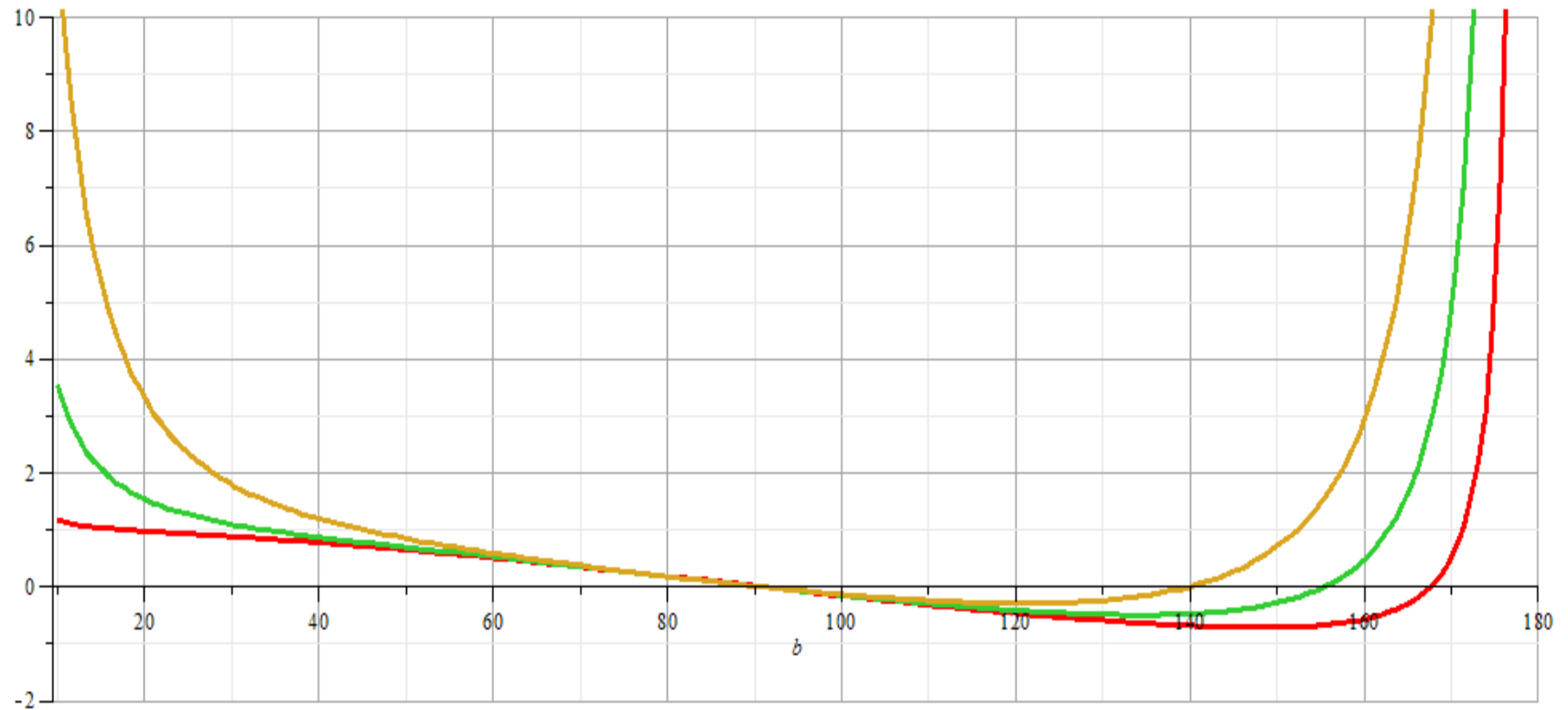
See also --

The rise and fall of spinning tops

American Journal of Physics **81**, 280 (2013); <https://doi.org/10.1119/1.4776195>

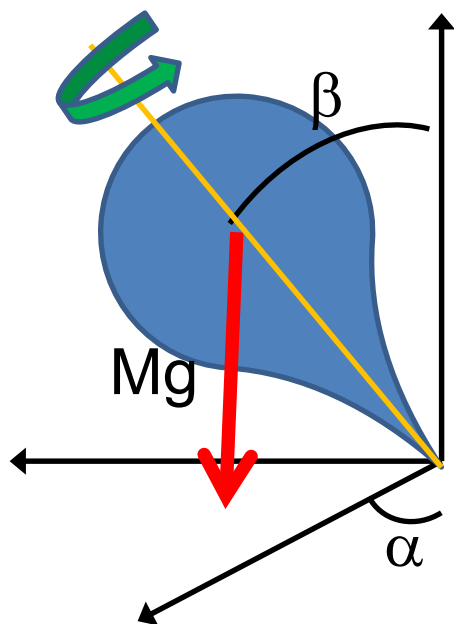
More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$



<https://drive.google.com/file/d/0B14RyYwpwSDNcXdxTWI3OExHX1k/view>





Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$\begin{aligned} p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} &= I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta \\ &= I_1 \dot{\alpha} \sin^2 \beta + p_\gamma \cos \beta \end{aligned}$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$