PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally in Olin 103

Plan for Lecture 24 – Chap. 5 (F &W)

Rotational motion

1. Torque free motion of a rigid body
2. Rigid body motion in body fixed frame
3. Conversion between body and inertial reference frames
4. Symmetric top motion

In this lecture, we continue our discussion of rigid body motion.

16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 5	Rigid body motion	#16	10/23/2020

Since you will be turning in your exams today, we will resume the homework assignments.

Due Wednesday

10/19/2020

PHY 711 – Homework # 15

Read Appendix A of **Fetter and Walecka**.

1. Assume that $a > 0$ and $b > 0$; use contour integration methods to evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{x^2 + b^2} dx.$$

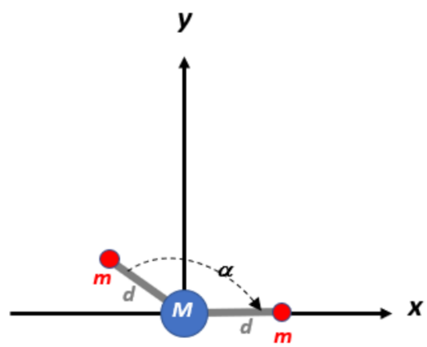
Note that you may use Maple, Mathematica, or other software to evaluate this integral, but full credit will be earned by using the contour integration methods.

This is a simple exercise using contour integration.

10/21/2020

PHY 711 – Homework # 16

Read Chap. 5 of **Fetter and Walecka**.

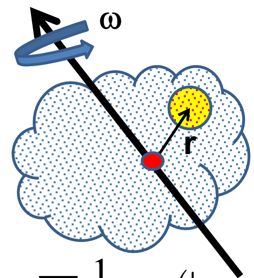


Due Friday

1. The figure above shows a rigid 3 atom molecule placed in the $x-y$ plane as shown.
   (a) Find the moment of inertia tensor in the given coordinate system placed at the center of mass M in terms of the masses, bond lengths d and angle α.
   (b) Find the principal moments I_1, I_2, and I_3 and the corresponding principal axes.
   (c) (Extra credit) Find the principal moments and axes for a coordinate system centered at the center of mass of the molecule.

This is an exercise to compute the moment of inertia tensor and diagonalize it.

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left| \boldsymbol{\omega} \times \mathbf{r}_p \right| \right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

Review of notions of rigid body motion.

Moment of inertia tensor

Matrix notation:

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

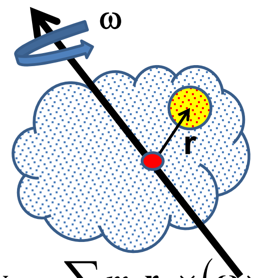
For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

In general there is a symmetric tensor which defines the moment of inertia. By rotating the coordinates about a fixed origin we can find the matrix in diagonal form.

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$

$$\mathbf{L} = \sum_p m_p \left[\boldsymbol{\omega}\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \mathbf{r}_p\left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)\right]$$

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

In addition to the kinetic energy, the angular momentum also can be expressed in terms of the moment of inertia tensor.

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

Time derivative: $\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

It is convenient to express the angular moment in terms of the principal moments .

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

When there is zero torque acting on the system, the angular velocity components are coupled through these Euler equations.

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\tilde{\omega}_3\left(I_3-I_2\right)=0$$

$$I_2\dot{\tilde{\omega}}_2+\tilde{\omega}_3\tilde{\omega}_1\left(I_1-I_3\right)=0$$

$$I_3\dot{\tilde{\omega}}_3+\tilde{\omega}_1\tilde{\omega}_2\left(I_2-I_1\right)=0$$

Solution for symmetric top -- $I_2=I_1$:

$$I_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\tilde{\omega}_3\left(I_3-I_1\right)=0$$

$$I_1\dot{\tilde{\omega}}_2+\tilde{\omega}_3\tilde{\omega}_1\left(I_1-I_3\right)=0$$

$$I_3\dot{\tilde{\omega}}_3=0 \qquad \Rightarrow \tilde{\omega}_3=\text{(constant)}$$

Define : $\Omega\equiv\tilde{\omega}_3\dfrac{I_3-I_1}{I_1}$

$$\dot{\tilde{\omega}}_1=-\tilde{\omega}_2\Omega$$

$$\dot{\tilde{\omega}}_2=\tilde{\omega}_1\Omega$$

The solution to the coupled angular velocity components is in general complicated, but simplifies when two of the principal moments are equal for a “symmetric top”.

Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\text{Solution :} \quad \tilde{\omega}_1(t) = A\cos(\Omega t + \varphi)$$

$$\tilde{\omega}_2(t) = A\sin(\Omega t + \varphi)$$

$$T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2 = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$
$$= I_1A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

Details of the solution for a symmetric top.

Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Now consider the case where all of the principal moments are unequal.

Euler equations for asymmetric top -- continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\quad \Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

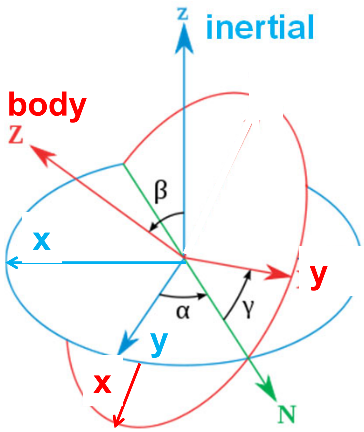
$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

We can find approximate stable solutions in certain cases.

Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

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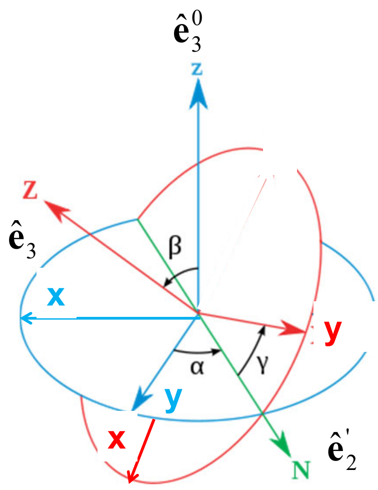
In order to consider motion of a rigid body more generally, in the presence of torque, it will be necessary to consider how to relate the body – fixed coordinates that diagonalize the moment of inertia tensor to another coordinate system which in general be an inertial coordinate system. Here again, we use ideas of Euler. This notation or it is equivalent is typically consistent with quantum mechanics text books.



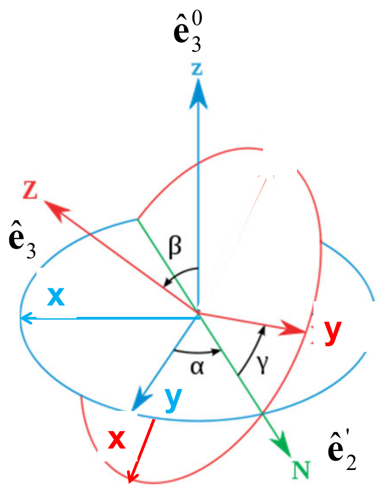
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

Euler said that the transformation of body-fixed → inertial frames can be accomplished in 3 steps and the corresponding angles are alpha, beta, and gamma. In this case, we want to express all results in the body fixed frame.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



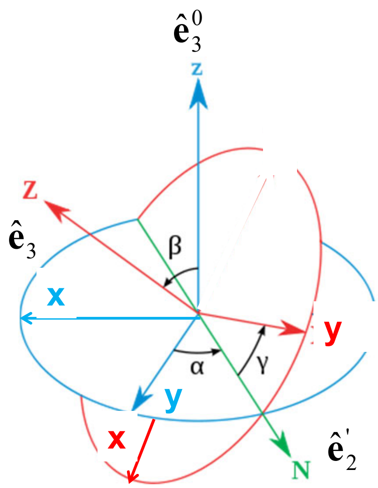
$$\hat{\mathbf{e}}_2' = \sin\gamma\,\hat{\mathbf{e}}_1 + \cos\gamma\,\hat{\mathbf{e}}_2$$

Matrix representation :

$$\hat{\mathbf{e}}_2' = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

We can express the angular velocities in terms of the time rate of change of the alpha, beta, and gamma Euler angles. We can also express the rotation axes in terms of the instantaneous Euler angles as well. Here is the transformation of the middle axis.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_3^0 = -\sin\beta\;\hat{\mathbf{e}}'_1 + \cos\beta\;\hat{\mathbf{e}}'_3$$
$$= -\cos\gamma\sin\beta\;\hat{\mathbf{e}}_1 + \sin\gamma\sin\beta\;\hat{\mathbf{e}}_2 + \cos\beta\;\hat{\mathbf{e}}_3$$

Matrix representation:

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

Here is the transformation of the inertial 3 axis.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\ \hat{\mathbf{e}}_3^0 + \dot{\beta}\ \hat{\mathbf{e}}_2^{'} + \dot{\gamma}\ \hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

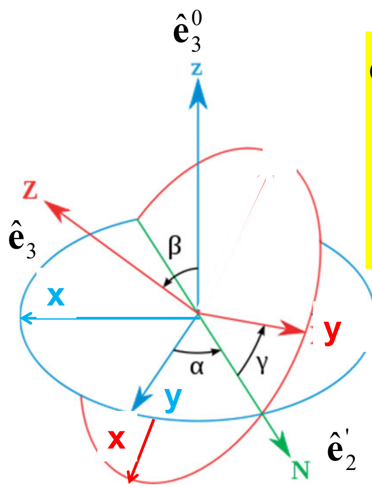
$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

Putting all of the transformations together, we now have expressions for the angular velocity components referenced to the body fixed frame.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Result to remember.

Rotational kinetic energy

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2$$

$$+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2$$

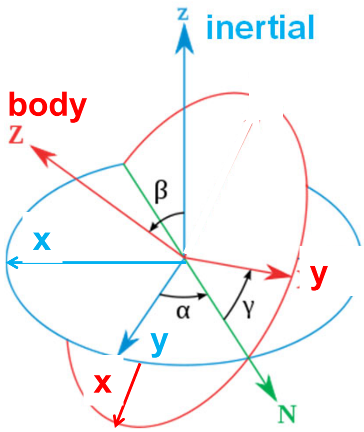
$$+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

If $I_1 = I_2$:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

General expression of the rotational kinetic energy and the special case of the symmetric top.

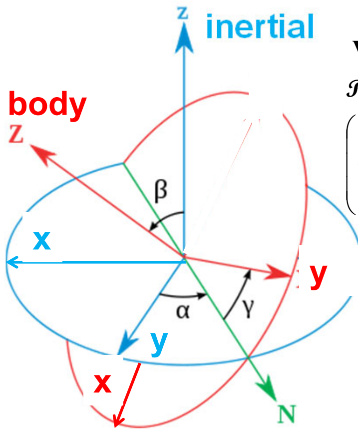
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

In addition to the dynamic transformation needed for rigid body mechanics, this formalism is more generally useful when relating coordinate systems of different orientations.

General transformation between rotated coordinates – Euler angles



$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

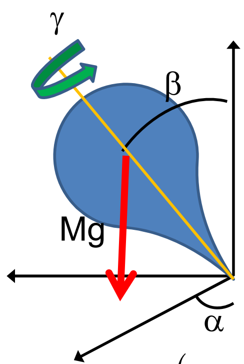
$$\mathcal{R} =$$

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

In general any transformation can be expressed in terms of the three Euler angles.

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2 - Mgl\cos\beta$$

Now consider the motion of a symmetric top in which the pivot point is fixed and torque is applied by gravity acting at the center of mass of the top. Here l denotes the distance of the pivot point to the center of mass.

$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) +$$

$$\frac{1}{2}I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]^2 - Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]\cos\beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]$$

$$E = \frac{1}{2}I_1\dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

$$L(\beta,\dot{\beta}) = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} - Mgl\cos\beta$$

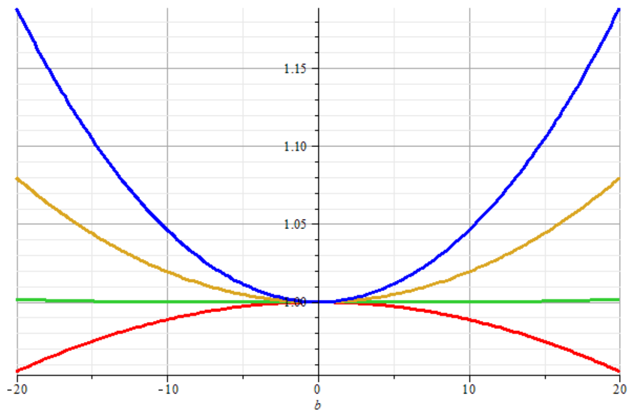
$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

Lagrangian and its solution.

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Stable/unstable solutions near β=0



Special case where top is spinning nearly vertically.

Suppose $\quad p_\alpha = p_\gamma \quad$ and $\quad \beta \approx 0$

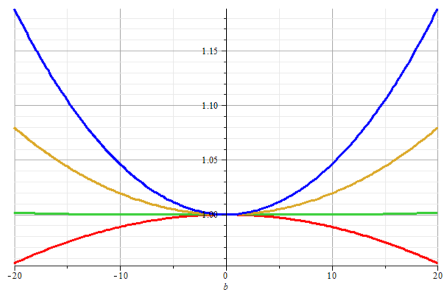
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma{}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma{}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

$\Rightarrow$ Stable solution if

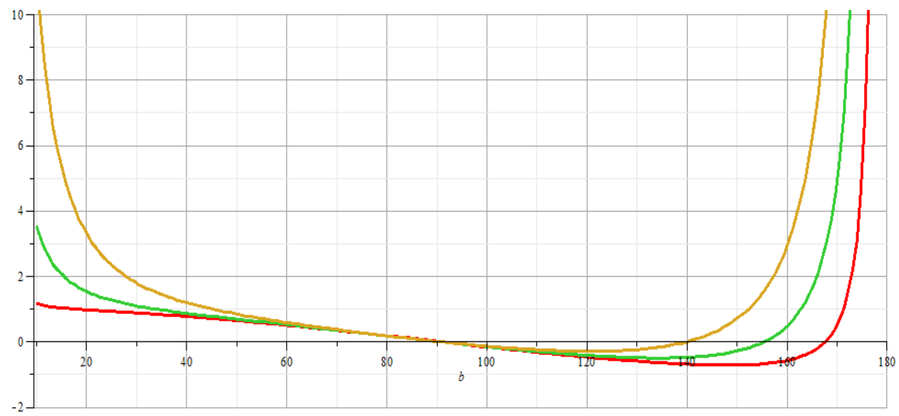
$$p_\gamma \geq \sqrt{4MglI_1}$$



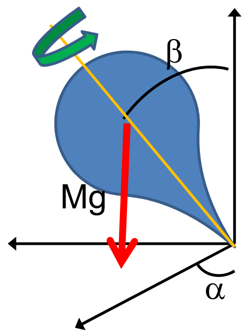
Approximate solution for that case.

More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$



For a spinning bicycle suspended by a rope, the beta angle can be greater than 90 degress



Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\cos\beta$$

$$= I_1\dot{\alpha}\sin^2\beta + p_\gamma\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

Summary of results.