PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103
Discussion for Lecture 25 – Chap. 8 (F & W)

Motions of elastic membranes

1. **Review of standing waves on a string**
2. **Standing waves on a two dimensional membrane.**
3. **Boundary value problems**

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 12 PM Friday

10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorm	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020

Friday – Review??

Monday -- Fluids

Thursday's Physics Online Colloquium

https://www.physics.wfu.edu/wfu-phy-news/seminars-2020-fall/

WAKE FOREST UNIVERSITY

Department of Physics

Colloquium

Oct. 22, 2020
at 4 PM

Ali Daraei

Graduate Student
Mentor, Dr. Martin Guthold
Physics Department
Wake Forest University, Winston-Salem, NC

"Intrinsically Unfolded Alpha-C Connector of Fibrinogen is a Major Contributor to the Mechanical Strength of Fibrin Fibers"

Fibrinogen is the key mechanical protein in blood coagulation since it is the building block of fibrin fibers, and these 100 nm thick fibers provide mechanical and structural stability to blood clots as they stem the flow of blood. In hemostasis, blood clots stop blood flow in the event of injury to blood vessels, and they are involved in the initiation of wound healing. In this case, they are beneficial – in fact, lifesaving – for the individual. On the other hand, in thrombosis, the aberrant formation of blood clots inside blood vessels causes serious diseases. For example, blood clots are the underlying pathology of myocardial infarction, ischemic strokes, deep vein thrombosis, and pulmonary embolism. In both scenarios, hemostasis and thrombosis, clots mechanically stop the flow of blood. The major structural

Your questions

From Nick –

1. For the lecture, where does this term come from? $\left(\nabla^2 + k^2\right)\rho(x,y) = 0$ For a standing wave, is the time derivative 0? Is there a systematic way to solve the boundary problems. I get that the solutions work, but it seems like it was just a good guess. How do we actually set up the system to solve?

From Gao –

1. About today's lecture, could you explain why standing waves have the form of R[(e-iwt)ρ(r)]?

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

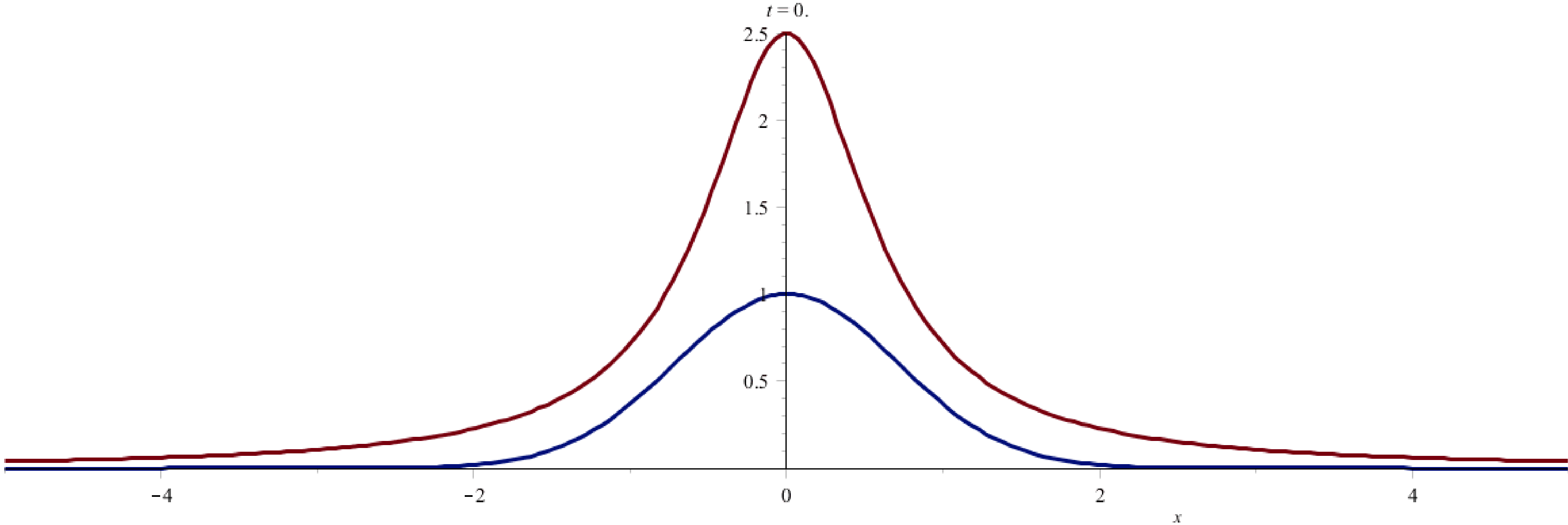
For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \dfrac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

with $\mu(0,t) = \mu(L,t) = 0.$

Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0 \qquad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$

Your questions -- About today's lecture, could you explain why standing waves have the form of R[(e-iwt)ρ(r)]?
For the lecture, where does this term come from? $\left(\nabla^2 + k^2\right)\rho(x,y) = 0$
For a standing wave, is the time derivative 0? Is there a systematic way to solve the boundary problems. I get that the solutions work, but it seems like it was just a good guess. How do we actually set up the system to solve?

Comment -- The basic idea is a “separable” partial differential equation which allows us to seek solutions that are products of functions of each variable alone.

Example -- for variable u and w:

Suppose $$\left(A(u)\frac{\partial^2}{\partial u^2} + B(w)\frac{\partial^2}{\partial w^2} \right) f(u,w) = 0$$

Example -- for variable u and w:

Suppose $$\left(A(u)\frac{\partial^2}{\partial u^2} + B(w)\frac{\partial^2}{\partial w^2} \right) f(u,w) = 0$$

Try to find a solution of the form:

$$f(u,w) = g(u)h(w)$$

Then, for all w and u :

$$h(w)A(u)\frac{\partial^2 g(u)}{\partial u^2} + g(u)B(w)\frac{\partial^2 h(w)}{\partial w^2} = 0$$

How can this work?

$$h(w)A(u)\frac{\partial^2 g(u)}{\partial u^2}+g(u)B(w)\frac{\partial^2 h(w)}{\partial w^2}=0$$

Suppose $A(u)\frac{d^2 g(u)}{du^2}=\alpha g(u)$ and $B(w)\frac{d^2 h(w)}{dw^2}=\beta h(w)$

Then $g(u)h(w)\left(\alpha+\beta\right)=0$

$\Rightarrow$ We have a solution for all u and w if $\beta=-\alpha$

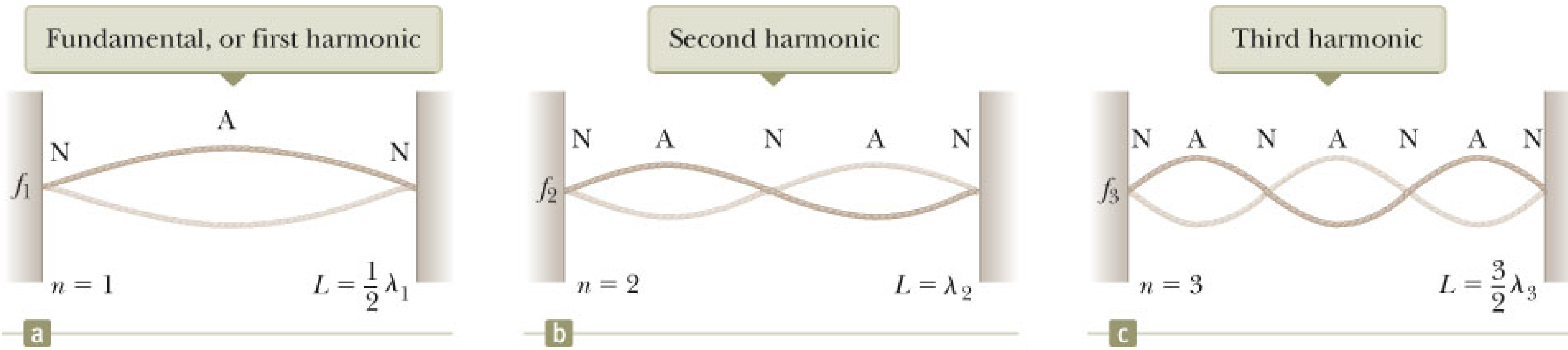
In our example of the wave equation:

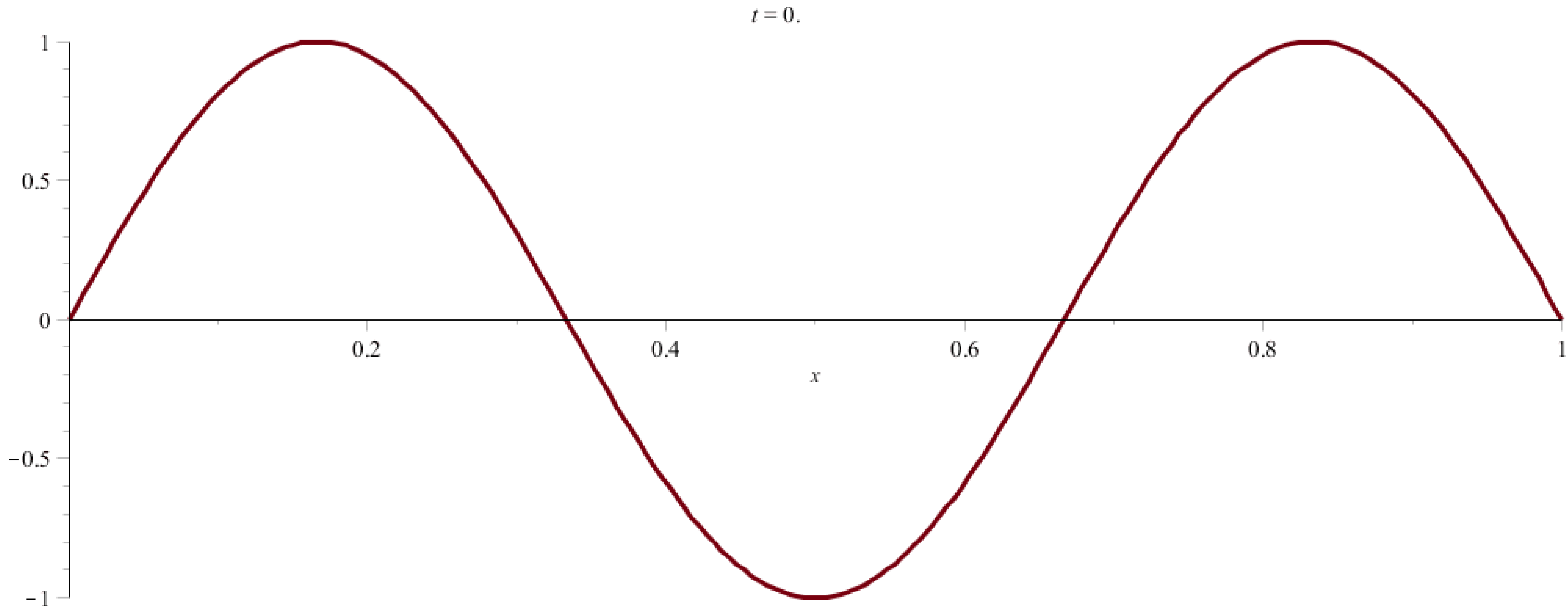
$$\frac{\partial^2 \mu}{\partial t^2}-c^2\frac{\partial^2 \mu}{\partial x^2}=0$$

$$\mu(x,t)=h(t)\rho(x)$$

$$\frac{d^2 h(t)}{dt^2}=-\omega^2 h(t) \qquad \frac{d^2 \rho(x)}{dx^2}=-k^2\rho(x)$$

$$k^2c^2=\omega^2$$


Fundamental, or first harmonic
A
N
N
f_1
$n = 1$
$L = \frac{1}{2}\lambda_1$
a
Second harmonic
N A N A N
f_2
$n = 2$
$L = \lambda_2$
b
Third harmonic
N A N A N A N
f_3
$n = 3$
$L = \frac{3}{2}\lambda_3$
c


t = 0.
1
0.5
0
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

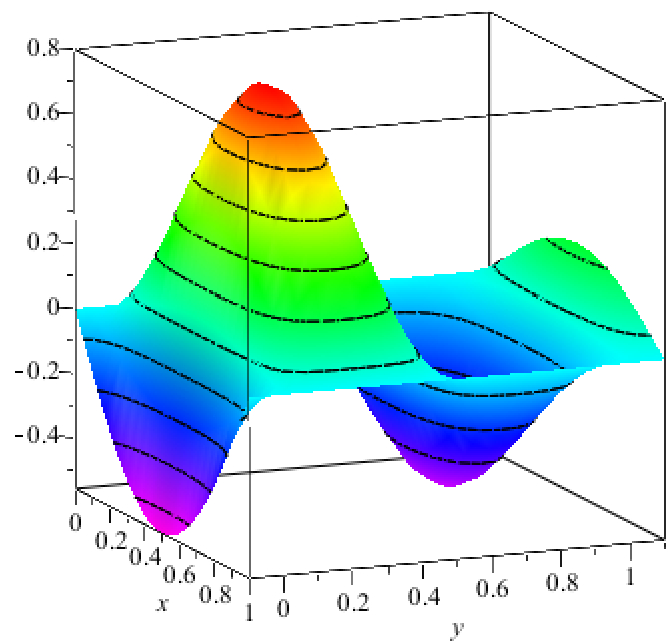
Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Note that here we are visualizing transverse waves. Longitudinal waves can also exist.

$\rho(x,y)$



In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(in Cartesian coordinates)}$$

Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\nabla u\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\partial u / \partial t\right)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\partial u / \partial x\right)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial\left(\partial u / \partial y\right)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

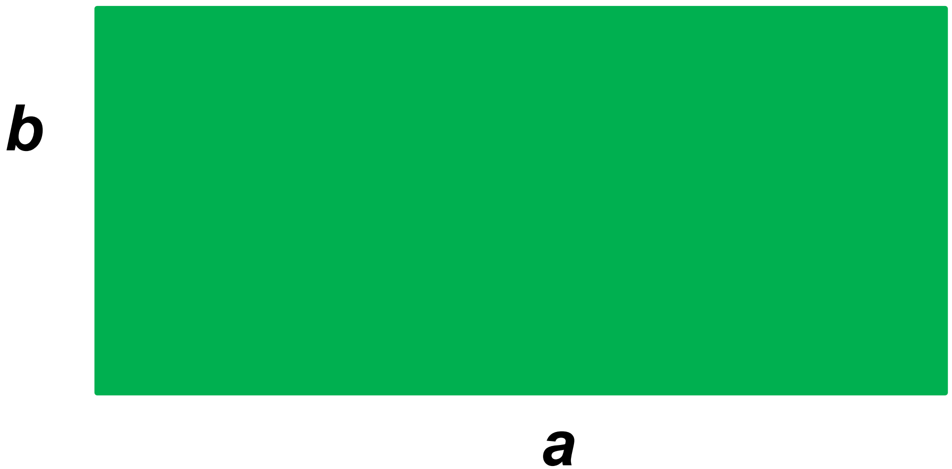
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:



Clamped boundary conditions:

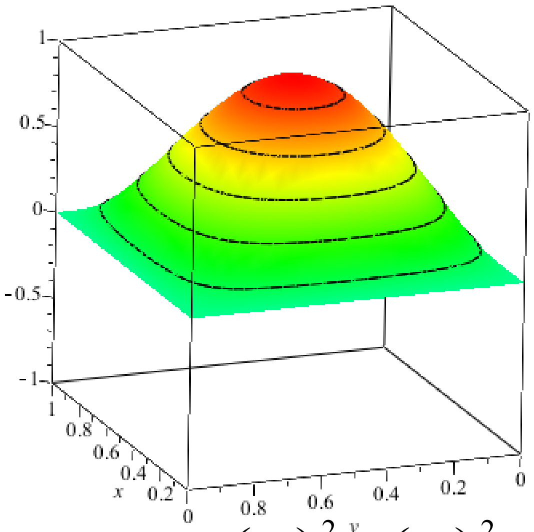
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

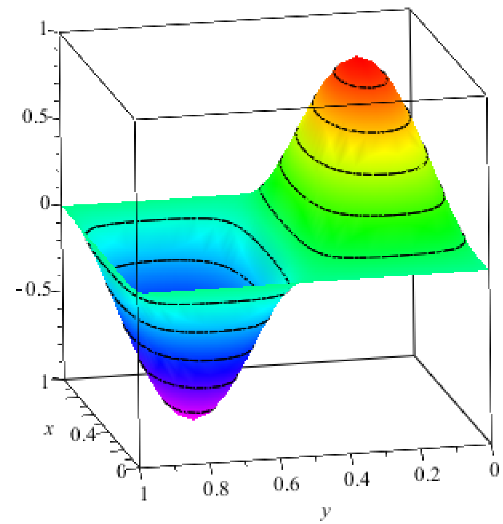
$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0$$

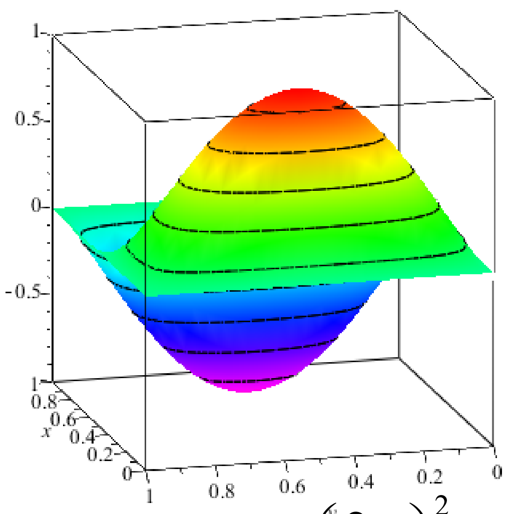
$$\text{where } k=\frac{\omega}{c}$$



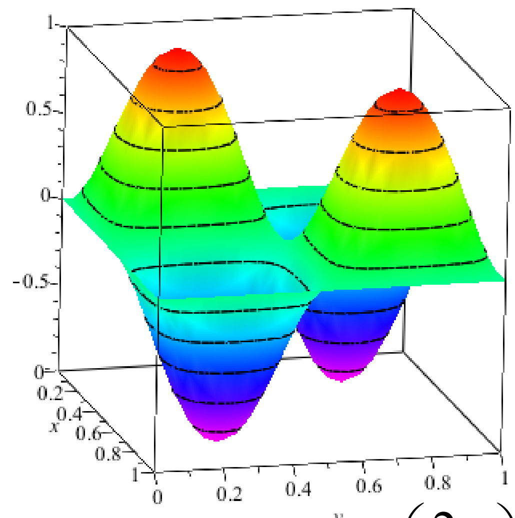
$$k_{11}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{12}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$



$$k_{21}^{\ 2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{22}^{\ 2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$$\tau\nabla u\big|_b = \kappa u\big|_b \qquad \text{represents bounded side constrained with spring}$$

$$\tau\nabla u\big|_b = 0 \qquad \text{represents "free" side}$$

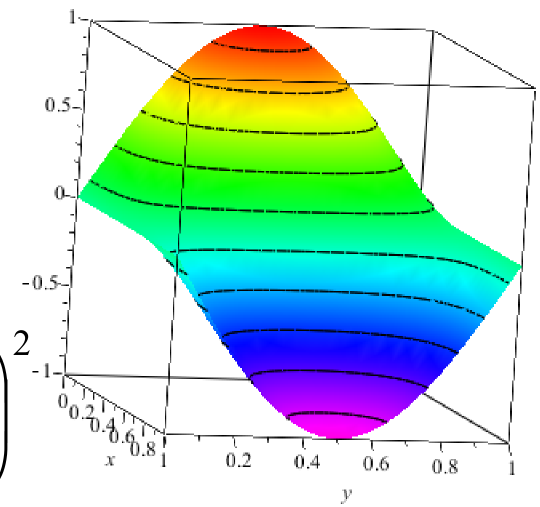
Mixed boundary conditions:

$$\rho(x,0) = \rho(x,b) = \frac{\partial\rho(0,y)}{\partial x} = \frac{\partial\rho(a,y)}{\partial x} = 0$$

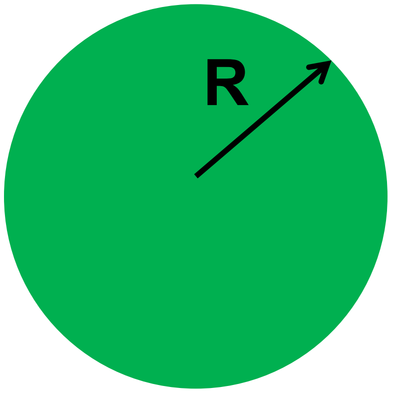
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$$\rho(R,\varphi)=0$$

$$\left(\nabla^2+k^2\right)\rho(r,\phi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$$\Rightarrow m=\text{integer}$$

Consider circular boundary -- continued

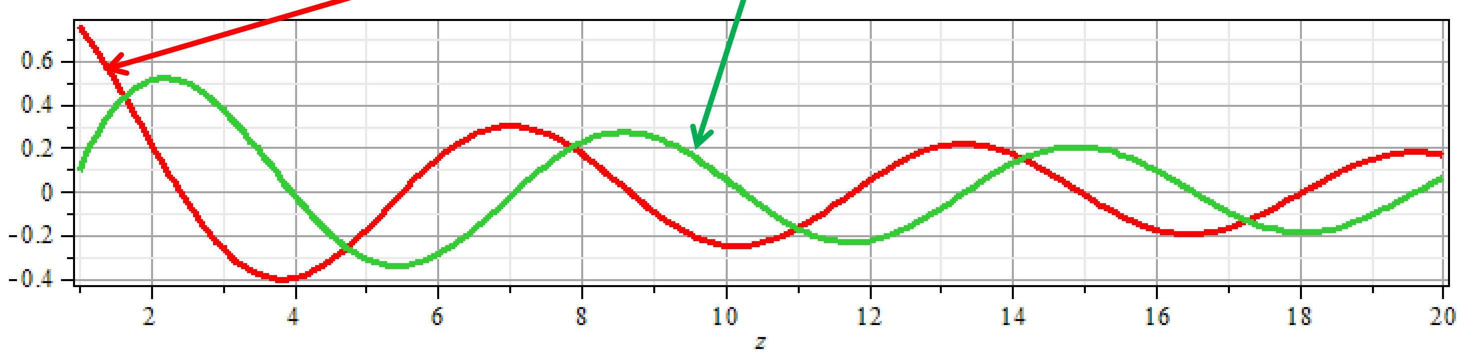
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

$$\text{Asending series:}\quad J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+m)!}\left(\frac{z}{2}\right)^{2j}$$

$$\text{Recursion relations:}\quad J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

$$\text{Asymptotic form:}\quad J_m(z) \xrightarrow[z>>1]{} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

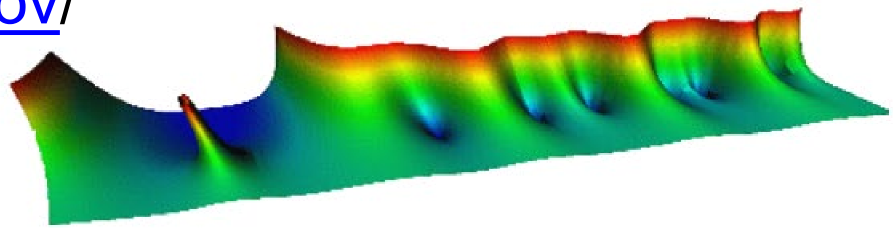
$$\text{Zeros of Bessel functions}\quad J_m(z_{mn}) = 0$$

$$m=0: \qquad z_{0n} = 2.406,\ 5.520,\ 8.654, \ldots$$

$$m=1: \qquad z_{1n} = 3.832,\ 7.016,\ 10.173, \ldots$$

$$m=2: \qquad z_{2n} = 5.136,\ 8.417,\ 11.620, \ldots$$

http://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 *DLMF Update; Version 1.0.9*
2014-04-25 *DLMF Update; Version 1.0.8; errata & improved MathML*
2014-03-21 *DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics*
2013-08-16 *Bille C. Carlson, DLMF Author, dies at age 89*
More news

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right) J_n(z)$$

$$-\frac{\left(\tfrac{1}{2}z\right)^{n}}{\pi} \sum_{k=0}^{\infty} \left(\psi(k+1)+\psi(n+k+1)\right) \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,(n+k)!},$$

Some properties of Bessel functions -- continued

Note: It is possible to prove the following

identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

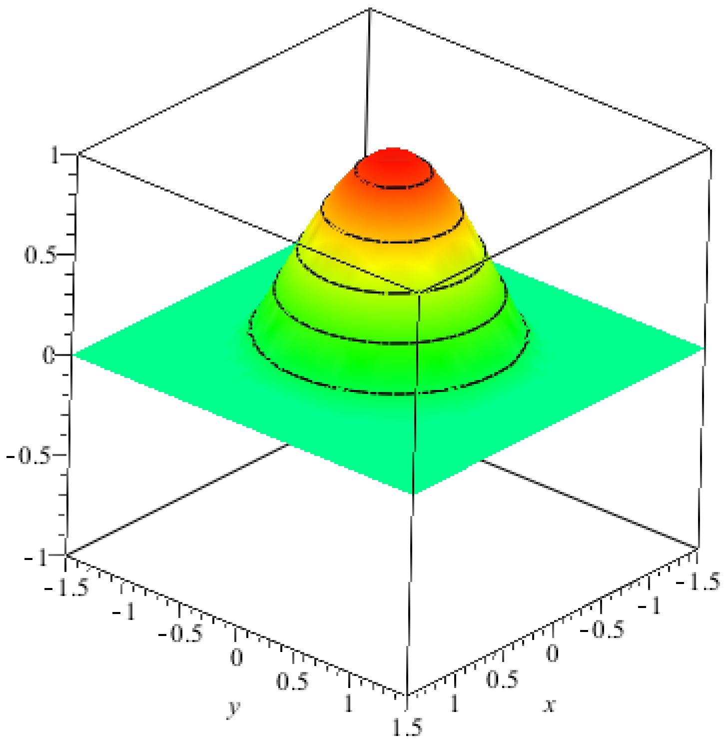
$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr = \frac{R^2}{2}\left(J_{m+1}\left(z_{mn}\right)\right)^2\delta_{nn'}$$

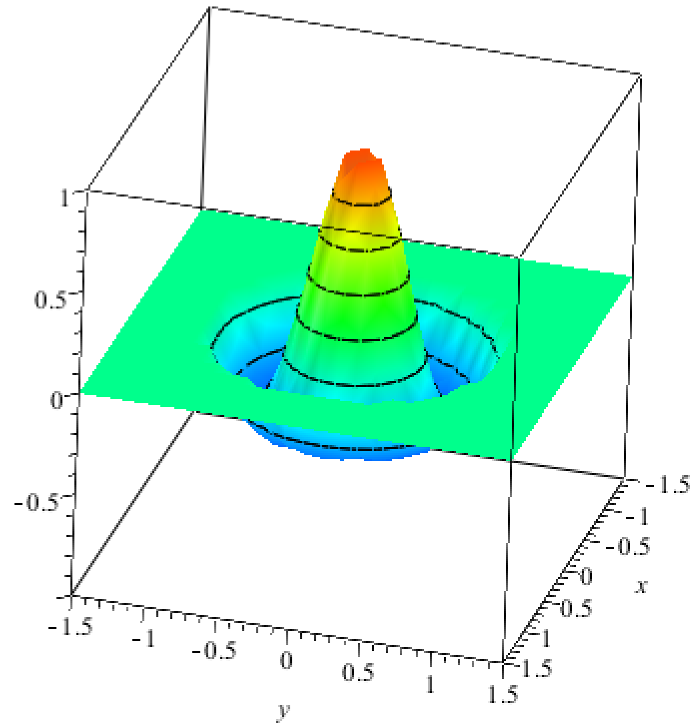
Returning to differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\ f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right) \qquad \rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$

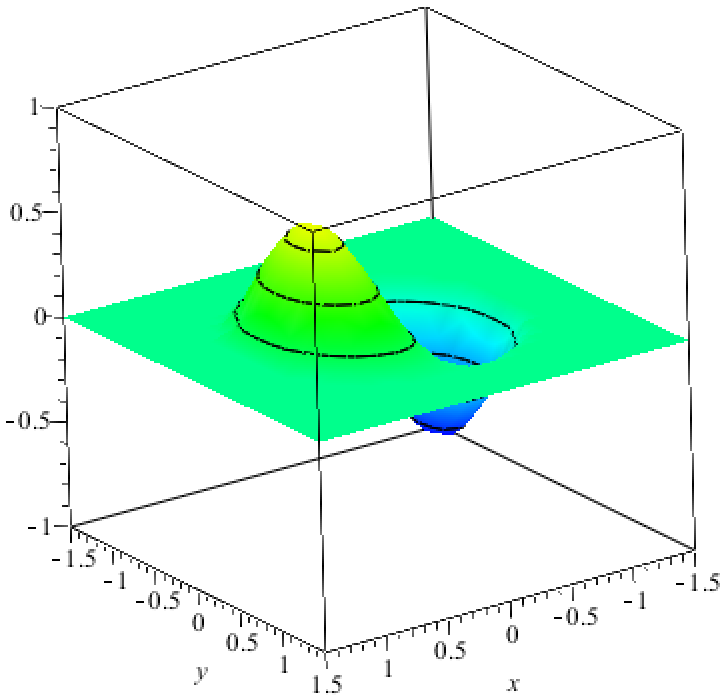




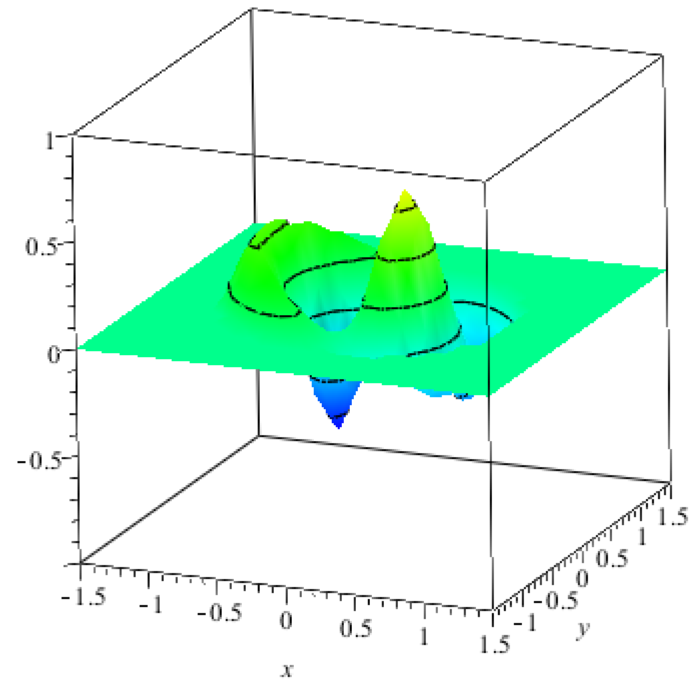
$$k_{01} = \frac{2.406}{R} \qquad k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$



$$k_{11} = \frac{3.832}{R}$$

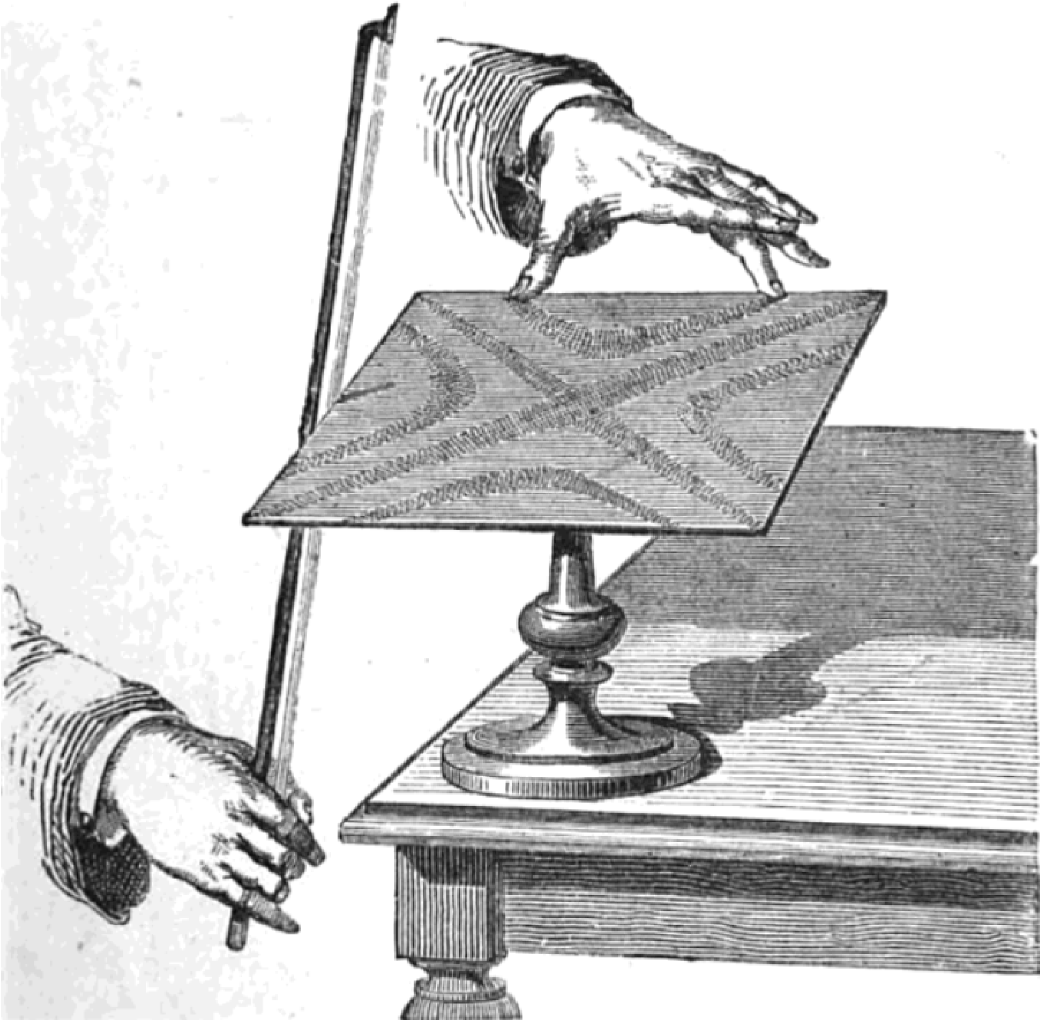


$$k_{12} = \frac{7.016}{R}$$

Ernst Chladni

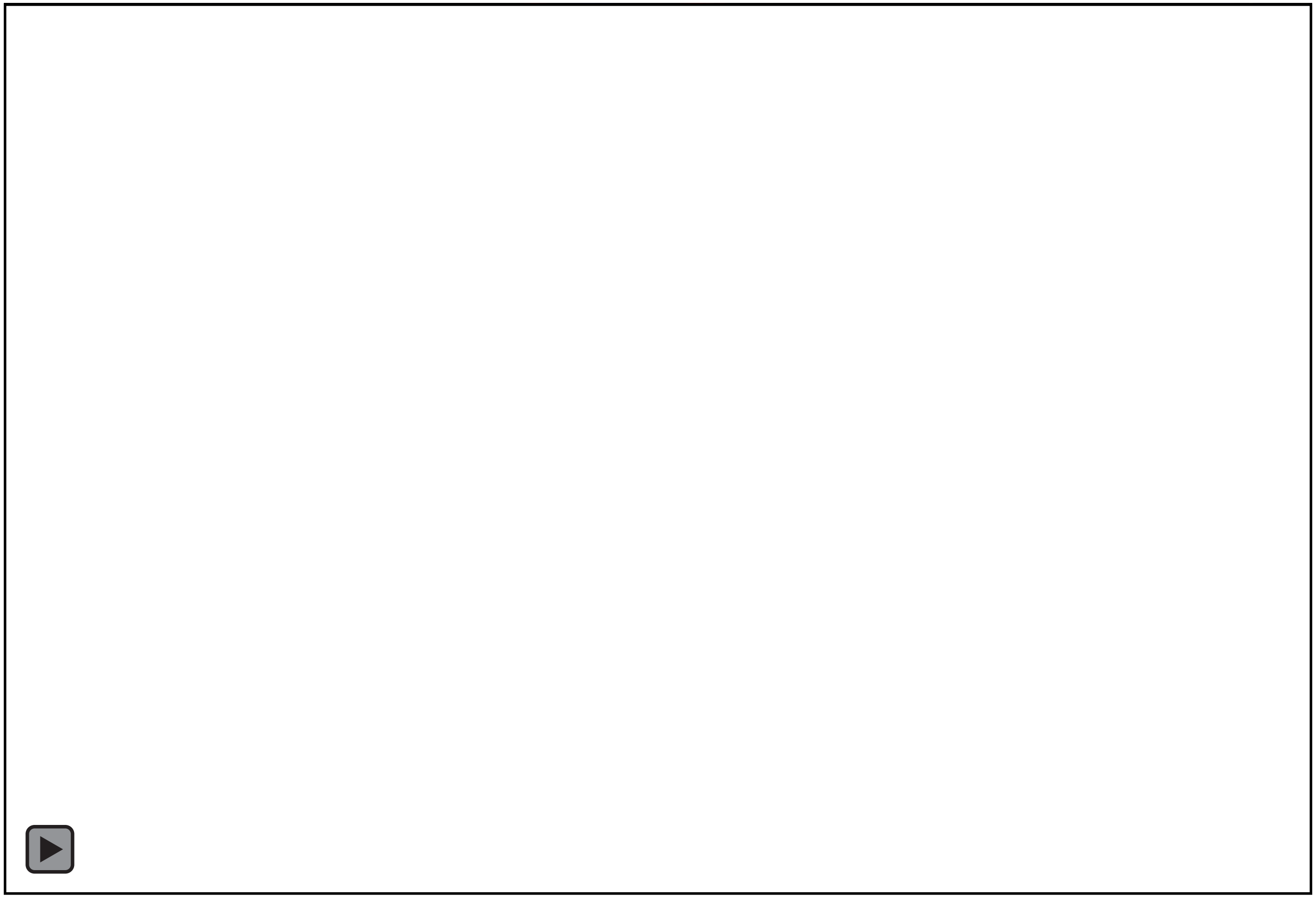
Ernst Chladni

Born	30 November 1756 Wittenberg, Electorate of Saxony in the Holy Roman Empire
Died	3 April 1827 (aged 70) Breslau, Province of Silesia in the Kingdom of Prussia, a part of the German Confederation
Nationality	German
Known for	Study of acoustics Chladni plates and figures Estimating the speed of sound Chladni's law Theory of meteorites' origins
	Scientific career
Fields	Physics

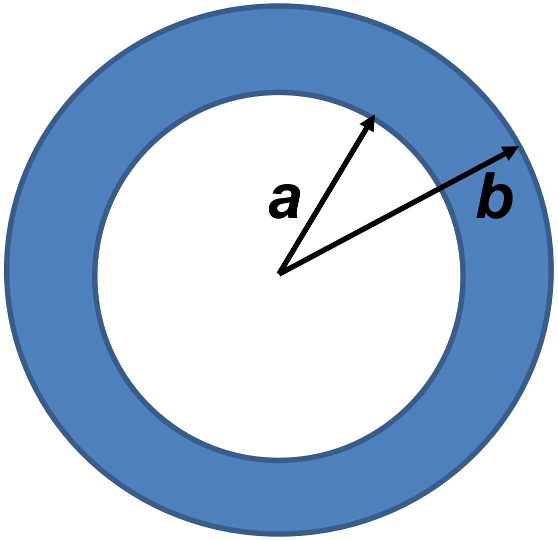


Demonstration with motor in the middle – (PASCO)

http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/



More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$\Rightarrow m = \text{integer}$$

Consider circular boundary -- continued

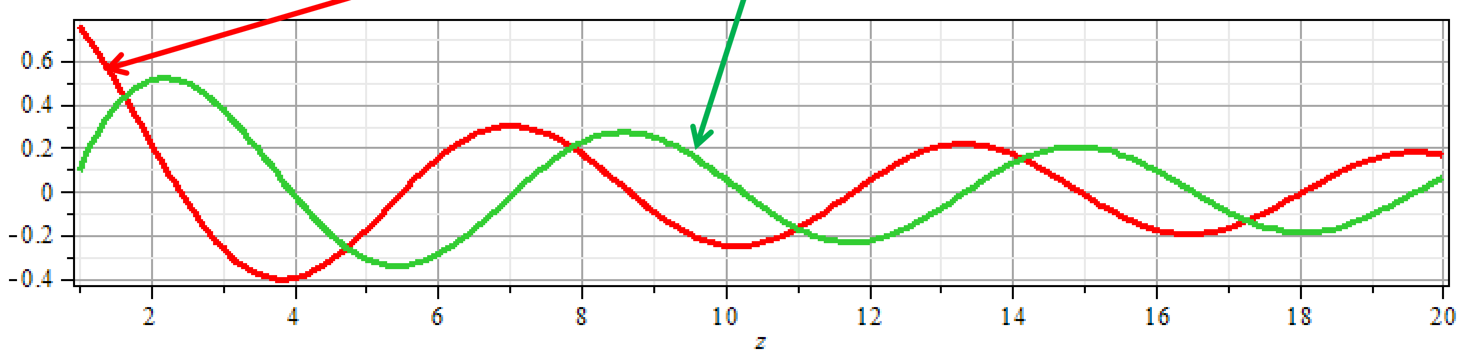
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

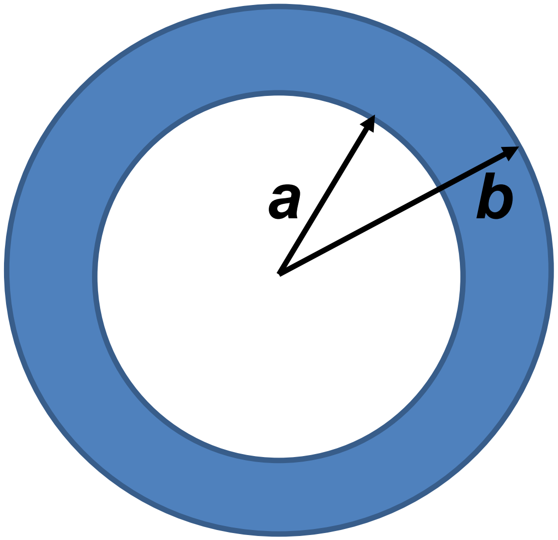
$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Normal modes of an annular membrane -- continued

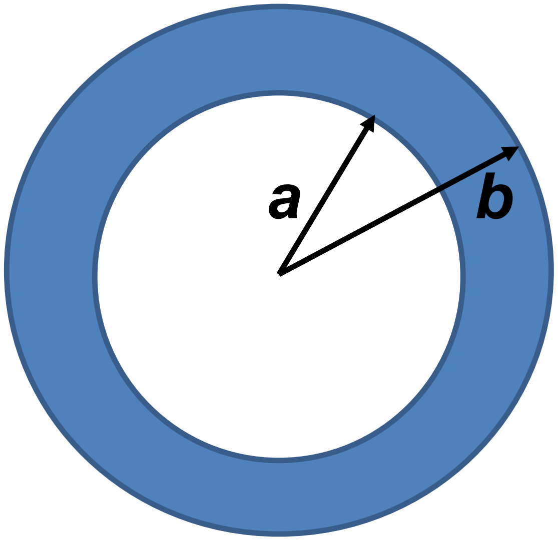


Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

Normal modes of an annular membrane -- continued



Boundary conditions:

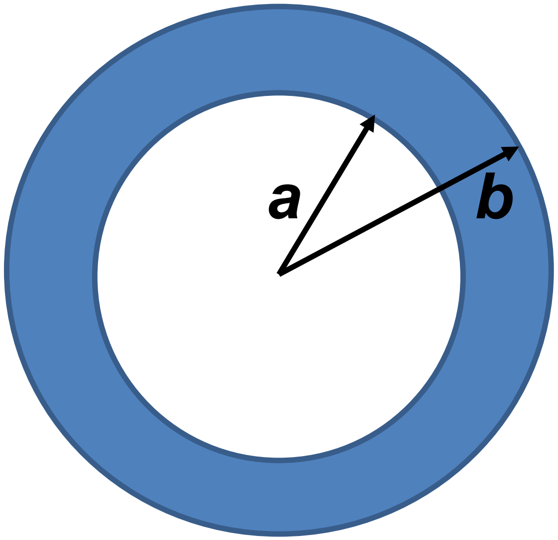
$f(a) = 0 \qquad f(b) = 0$

$$AJ_m(ka) + BN_m(ka) = 0$$
$$AJ_m(kb) + BN_m(kb) = 0$$

$\Rightarrow$ 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad \text{(transcendental equation for } k\text{)}$$

Normal modes of an annular membrane -- continued



Boundary conditions:

$$f(a)=0 \qquad\qquad f(b)=0$$

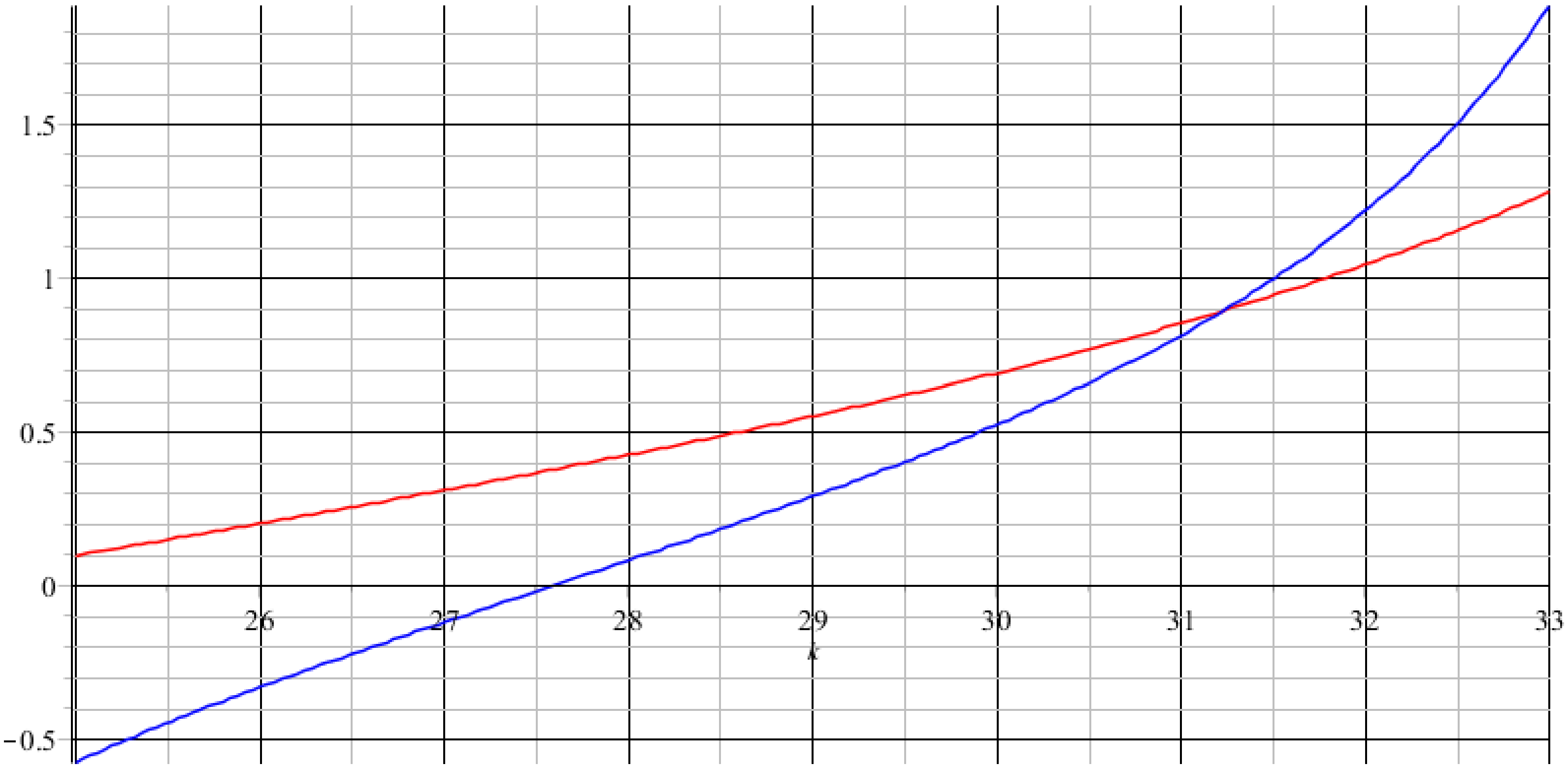
$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)}$$ -- in terms of solution k_{mn} :

$$f(r)=A\left(J_m(k_{mn}r)-\frac{J_m(k_{mn}a)}{N_m(k_{mn}a)}N_m(k_{mn}r)\right)$$

Analysis for *m=0* and a=0.1, b=0.2:

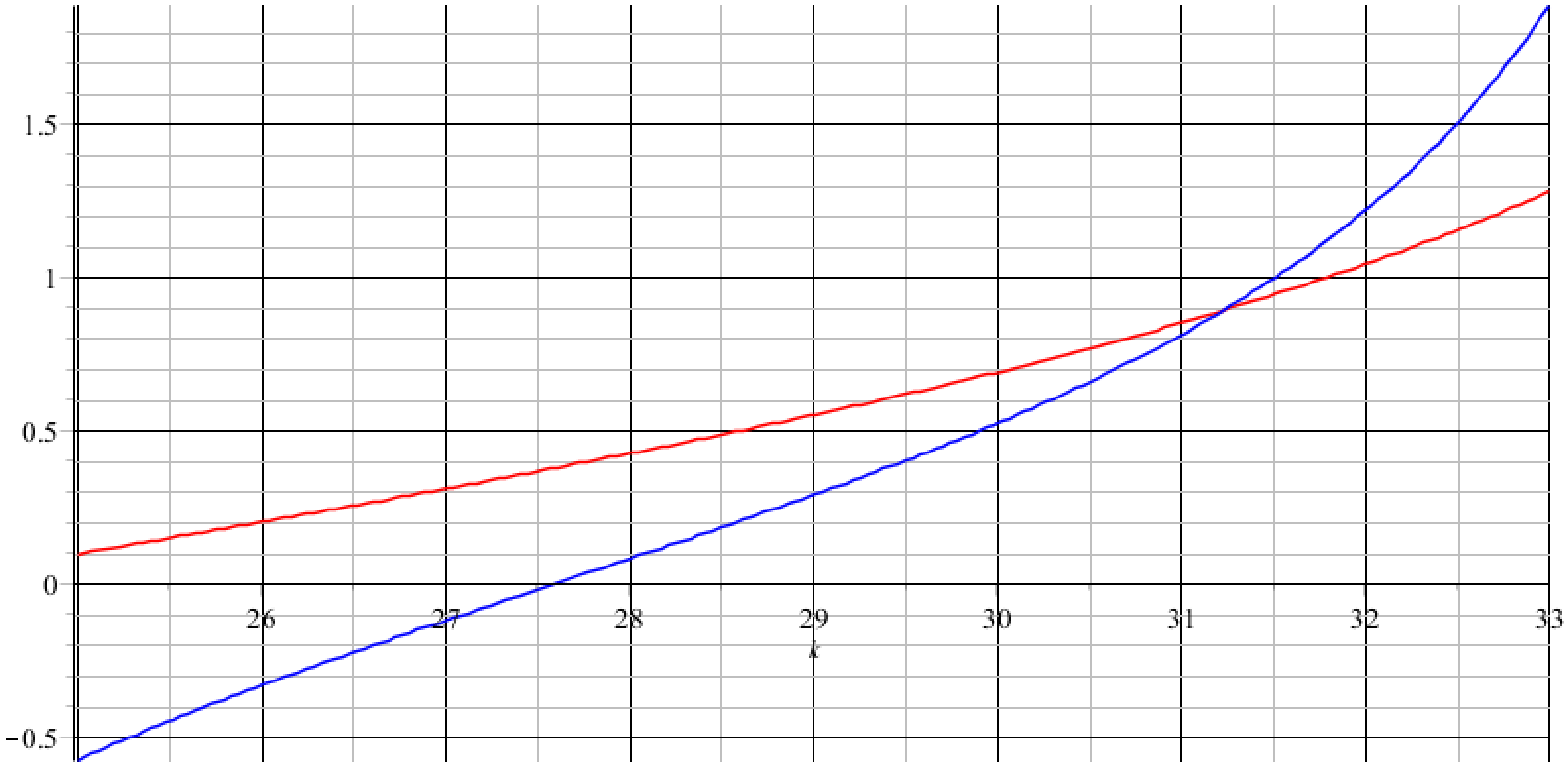
=

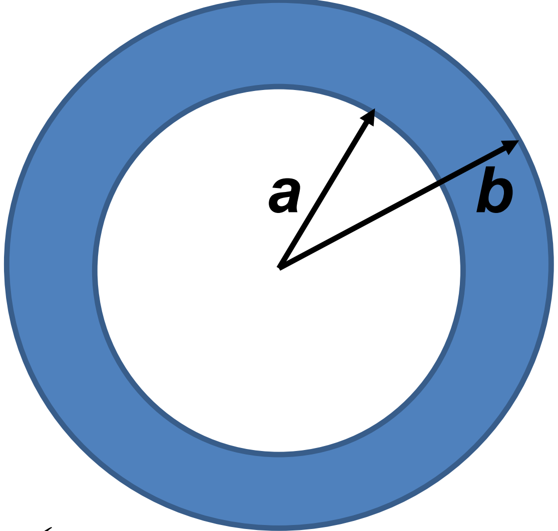
$$> plot\left(\left\{\frac{-\mathrm{BesselJ}(0,\,0.1\cdot k)}{\mathrm{BesselY}(0,\,0.1\cdot k)},\ \frac{-\mathrm{BesselJ}(0,\,0.2\cdot k)}{\mathrm{BesselY}(0,\,0.2\cdot k)}\right\},\ k=25\,..33,\ color=[red,\ blue]\right);$$



$$> fsolve\left(\frac{-\mathrm{BesselJ}(0,\ 0.1\cdot k)}{\mathrm{BesselY}(0,\ 0.1\cdot k)} = \frac{-\mathrm{BesselJ}(0,\ 0.2\cdot k)}{\mathrm{BesselY}(0,\ 0.2\cdot k)},\ k,\ 30\ ..33 \right);$$

31.23030920





$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \qquad k_{01} = 31.230309$$